

The linearised conformal Einstein field equations: Petrov-type D backgrounds

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The Einstein equations under conformal transformations?

- Penrose proposal in a nut-shell: $g = \Xi^2 \tilde{g}$
- What are the equations for the unphysical metric g ?
- Einstein field equations in vacuum

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}$$

Are not conformally invariant!

- A calculation shows that

$$g_{ab} = \Xi^2 \tilde{g}_{ab}, \quad \implies$$

$$R_{ab} = \tilde{R}_{ab} - 2\Xi^{-1} \nabla_a \nabla_b \Xi - g_{ab} (\Xi^{-1} \nabla^c \nabla_c \Xi - 3\Xi^{-2} \nabla_c \Xi \nabla^c \Xi),$$

where R_{ab} and ∇_a are associated with g_{ab} .

- Formally **singular** at $\Xi = 0$!

$$g_{ab} = \Xi^2 \tilde{g}_{ab}, \quad \implies$$

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Formulation classification answer the question: What to do Ξ ?...

- Ξ given: **Hyperboloidal-Singular-Scri-fixed. Singular equation for g**
(Formulation and numerics: Zenginoglu, Vañó, Husa, Macedo ...)
- There's no Ξ : Hiperboloidal dual-foliation: Non-conformal rescaling. Singular terms. (Formulation: Hilditch, G. , Duarte, Feng. Numerics: Vañó, Peterson)
- Solve for Ξ : Elliptic-hyperbolic formulation. Formally singular equations. Impose extra conditions for regular limits. (Rinne-Moncrief, Bardeen-Sarbach-Buchman)
- **Evolve Ξ : Friedrich CEFEs: Formally regular.** (Formulation: Friedrich, Apps in Math-Rel.: Valiente, Lübbe, G. ..., Numerics: Frauendiener, Stevens, Thwala)

$$\nabla_a \nabla_b \Xi = -\frac{\Xi}{2} R_{ab} + \dots$$

The **conformal** Einstein field equations (H. Friedrich 81)

Equations:

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab},$$

$$\nabla_a s = -L_{ac} \nabla^c \Xi,$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = d^a{}_{bcd} \nabla_a \Xi,$$

$$\nabla_a d^a{}_{bcd} = 0$$

Fields: $(\Xi, s, L_{ab}, d^a{}_{bcd})$

$$L_{ab} := \frac{1}{2} R_{ab} - \frac{1}{12} R g_{ab}, \quad (\text{Schouten tensor})$$

$$s := \frac{1}{4} \nabla_a \nabla^a \Xi + \frac{1}{24} R \Xi, \quad (\text{Friedrich's scalar})$$

$$d^a{}_{bcd} := \Xi^{-1} C^a{}_{bcd}. \quad (\text{Rescaled Weyl tensor})$$

$$6\Xi s - 3\nabla_c \Xi \nabla^c \Xi = \lambda$$

Hyperbolic reduction of the CEFs

- Metric:

$$R_{\mu\nu} = 2L_{\mu\nu} + \frac{1}{6} R g_{\mu\nu}$$

- e.g. (generalised) harmonic gauge $\square x^\mu = F^\mu \rightsquigarrow$
 $g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} = \dots$

- Tetrads: Cartan structure equations

Extended conformal Einstein field equations XCEFE

- Non-metric connections (Weyl-connection): $\hat{\nabla}_a \tilde{g}_{bc} = -2\tilde{f}_a \tilde{g}_{bc}$
- Advantage: Geo-conf $\rightsquigarrow \hat{\Xi}(\tau) = 0 \implies \Xi(\tau)$ "known".
- Obs. $\varphi^* g = \Xi^2 \tilde{g}$, i.e., $x^\mu = x^\mu(\tilde{x}^\mu)$ non-explicit.

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Evo. Ξ

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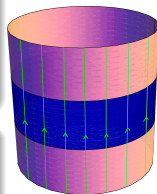


Image credits: Conf. Methods in GR book by J. Valiente

Hyperboloidal Approach

Non-linear hyperboloidal

Hyp. fols. provide **formally singular** (Ξ^{-1} type terms) alternatives to the CEFs (Vaño, Hilditch, Zenginoglu, Husa...)

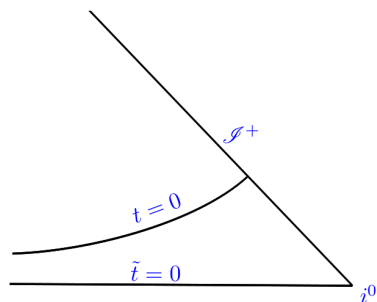
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- $\tilde{t} = 0$ Cauchy, $t = 0$ Hyp.
- $\tilde{t} = t + H(r)$, $\tilde{r} = \frac{r}{\Omega(r)}$

Linear hyperboloidal

Hyp. fols. provide **formally regular** (without Ξ^{-1} type terms) compactified (in hyp. r coord) representation of linear perts of BHs (Zenginoglu, Panosso-Macedo, Jaramillo, Pound, Leather, Minucci, Boyanov, Cardoso, Meneses Rojas, Al Sheik, Warburton...)

- Accurate QNMs
- Pseudospectrum
- Self-force
- ...prolific!...
- Effective: starts from LINEAR (physical/uncompactified) eq.
- Coord. based: Compactifies via Hyp. coords-based-expressions



EFE

Compactification

\exists given

\exists evd

Conformally sing.
Einstein
formally singular

CEFE
formally regular

linearisation

Teuk

Coord.-based (Hyperboloidal)

Compactification

Hyp Teuk
formally regular

EFE

Compactification

\exists given

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Conformally sing.
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linearisation

NR Hyperboloidal
Community

Teuk

Coord.-based (Hyperboloidal)

compactification

Hyp Teuk
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EFE

Compactification

\exists given

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Conformally sing.
Einstein
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\neq

CEFE
formally regular

linearisation

Standard effective approach

Coord.-based (Hyperboloidal)

compactification

Teuk

Hyp Teuk
formally regular



Distant cousins?
Missing link!

EFE

Compactification

\exists given

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Conformally sing.
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\neq

CEFE
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linearisation

linearisation

Conf-Teuk (☺)

Hyp coords
 f -adapted

ρ - β adapted
coords

Coord.-based (Hyperboloidal)

Hyp Teuk
formally regular

ρ -Cyl (like) Teuk

Teuk

Compactification

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Hyp coords
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Coord.-based (Hyperboloidal)

Hyp Teuk
formally regular

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Teuk

Compactification

Metric CEFEs and their linearisation

- Non linear: (Paetz 13, Carranza Hursit Valiente 19). CEFE analogue of EFE in (generalised) harmonic gauge
 $\square x^\mu = F^\mu (= 0)$.
- Linearisation:
- á la Penrose (non-metric):

$$\overset{\circ}{\nabla}_{A'}{}^A \check{\phi}_{ABCD} = 0 \text{ for (conf-) Mink}$$

- á la Einstein/deDonder (metric)...

$$g = \overset{\circ}{g} + \check{g}, \quad \Xi = \overset{\circ}{\Xi} + \check{\Xi}$$

(Feng & G. 23) CEFE analogue of linear perts. in (generalised) Lorentz gauge.

$$\overset{\circ}{\nabla}^\nu \check{g}_{\mu\nu} = f_\mu (= 0)$$

$$\overset{\circ}{\square} \check{g}_{\mu\nu} = S_{(\mu\nu)}^g$$

$$\overset{\circ}{\square} \check{\Xi} = S^\Xi$$

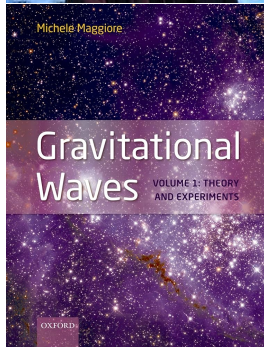
$$\overset{\circ}{\square} \check{s} = S^s$$

$$\overset{\circ}{\square} \check{\Phi}_{\mu\nu} = S_{(\mu\nu)}^\Phi$$

$$\overset{\circ}{\square} \check{d}_{\mu\nu\alpha\beta} = S_{[\mu\nu][\alpha\beta]}^d (= 0 \text{ for [conf-] Mink}).$$



Michele Maggiore



If $\dot{d}_{\mu\nu\alpha\beta} = 0 \implies$ decoupling of Bianchi system!

If $\dot{d}_{\mu\nu\alpha\beta} \neq 0 \implies$ coupling with the **perturbations** $\check{\Phi}_{\mu\nu}$, $\check{g}_{\mu\nu}$, $\check{\Xi}$ (dynamical quantities satisfying their own evolution equations!)...

$$\begin{aligned}
 S_{\mu\nu\alpha\beta}^d &= \frac{1}{2} \mathring{R} \check{d}_{\mu\nu\alpha\beta} - (4\check{\Phi}^{\delta\sigma} \check{g}_{\delta\sigma} + \frac{1}{12} \mathring{R} \check{g}^{\delta}_{\delta}) \dot{d}_{\mu\nu\alpha\beta} \\
 &- 2\check{\Phi}^{\delta\sigma} \check{g}_{\alpha\delta} \dot{d}_{\mu\nu\beta\sigma} - 6\check{\Phi}_{\alpha}^{\delta} \check{g}_{\delta}^{\sigma} \dot{d}_{\mu\nu\beta\sigma} + 4\check{\Phi}_{\alpha}^{\delta} \dot{d}_{\mu\nu\beta\delta} - \frac{1}{6} \mathring{R} \check{g}_{\alpha}^{\delta} \dot{d}_{\mu\nu\beta\delta} \\
 &+ 2\check{\Phi}_{\alpha}^{\delta} \check{g}^{\sigma}_{\sigma} \dot{d}_{\mu\nu\beta\delta} - 2\check{\Xi} \check{d}_{\alpha}^{\delta} \beta^{\sigma} \dot{d}_{\mu\nu\delta\sigma} - 2\check{\Xi} \dot{d}_{\alpha\delta\beta\sigma} \check{d}_{\mu\nu}^{\delta\sigma} - 2\check{\Xi} \dot{d}_{\alpha\delta\beta\sigma} \dot{d}_{\mu\nu}^{\delta\sigma} \\
 &+ 4\check{\Phi}_{\mu}^{\delta} \dot{d}_{\nu\delta\alpha\beta} - \frac{1}{6} \mathring{R} \check{g}_{\mu}^{\delta} \dot{d}_{\nu\delta\alpha\beta} + 2\check{\Phi}_{\mu}^{\delta} \check{g}^{\sigma}_{\sigma} \dot{d}_{\nu\delta\alpha\beta} - 8\check{\Xi} \check{d}_{\mu}^{\delta} \alpha^{\sigma} \dot{d}_{\nu\delta\beta\sigma} \\
 &- 4\check{\Xi} \dot{d}_{\mu}^{\delta} \alpha^{\sigma} \dot{d}_{\nu\delta\beta\sigma} + 2\check{\Xi} \check{g}^{\delta\sigma} \dot{d}_{\beta\delta\sigma\lambda} \dot{d}_{\mu\nu\alpha}^{\lambda} + 2\check{\Xi} \check{g}^{\delta\sigma} \dot{d}_{\mu}^{\lambda} \alpha_{\beta} \dot{d}_{\nu\delta\sigma\lambda} - 6\check{\Phi}_{\mu}^{\delta} \check{g}_{\delta}^{\sigma} \dot{d}_{\nu\sigma\alpha\beta} \\
 &- 2\check{\Phi}^{\delta\sigma} \check{g}_{\mu\delta} \dot{d}_{\nu\sigma\alpha\beta} + 4\check{\Xi} \check{g}^{\delta\sigma} \dot{d}_{\alpha\sigma\beta\lambda} \dot{d}_{\mu\nu\delta}^{\lambda} + 4\check{\Xi} \check{g}^{\delta\sigma} \dot{d}_{\mu}^{\lambda} \alpha_{\delta} \dot{d}_{\nu\lambda\beta\sigma} + 4\check{\Xi} \check{g}^{\delta\sigma} \dot{d}_{\mu\delta\alpha}^{\lambda} \dot{d}_{\nu\sigma\beta\lambda} \\
 &- \check{\Xi} \check{g}^{\delta}_{\delta} \dot{d}_{\alpha\sigma\beta\lambda} \dot{d}_{\mu\nu}^{\sigma\lambda} - 2\check{\Xi} \check{g}^{\delta}_{\delta} \dot{d}_{\mu}^{\sigma} \alpha^{\lambda} \dot{d}_{\nu\sigma\beta\lambda} - 2\mathring{\nabla}_{\alpha} \check{g}^{\delta\sigma} \mathring{\nabla}_{\sigma} \dot{d}_{\mu\nu\beta\delta} - 2\mathring{\nabla}_{\mu} \check{g}^{\delta\sigma} \mathring{\nabla}_{\sigma} \dot{d}_{\nu\delta\alpha\beta} \\
 &+ \check{g}^{\delta\sigma} \mathring{\nabla}_{\sigma} \mathring{\nabla}_{\delta} \dot{d}_{\mu\nu\alpha\beta} + 2\mathring{\nabla}_{\delta} \dot{d}_{\mu\nu\beta\sigma} \mathring{\nabla}^{\sigma} \check{g}_{\alpha}^{\delta} - 2\mathring{\nabla}_{\sigma} \dot{d}_{\mu\nu\beta\delta} \mathring{\nabla}^{\sigma} \check{g}_{\alpha}^{\delta} - \mathring{\nabla}_{\alpha} \dot{d}_{\mu\nu\beta\sigma} \mathring{\nabla}^{\sigma} \check{g}^{\delta}_{\delta} \\
 &- \mathring{\nabla}_{\mu} \dot{d}_{\nu\sigma\alpha\beta} \mathring{\nabla}^{\sigma} \check{g}^{\delta}_{\delta} - 2\mathring{\nabla}_{\sigma} \dot{d}_{\mu\nu\alpha\beta} \mathring{\nabla}^{\sigma} \check{g}^{\delta}_{\delta} + 2\mathring{\nabla}_{\delta} \dot{d}_{\nu\sigma\alpha\beta} \mathring{\nabla}^{\sigma} \check{g}_{\mu}^{\delta} - 2\mathring{\nabla}_{\sigma} \dot{d}_{\nu\delta\alpha\beta} \mathring{\nabla}^{\sigma} \check{g}_{\mu}^{\delta}.
 \end{aligned}$$

CEFEs in NP form (Valiente, Hilditch, Zhao 20)

Schematically:

$$\nabla\nabla\Xi \simeq \nabla\Xi \times \Gamma + \Xi \times \Phi + \Xi R + s$$

$$\nabla s \simeq R \times \nabla\Xi + \Phi \times \nabla\Xi$$

$$\nabla\Phi + \nabla R \simeq \Phi \times \Gamma + \Gamma \times \phi$$

$$\nabla\phi \simeq \Gamma \times \phi$$

$$\nabla\Gamma \simeq R + \Phi + \Xi\phi + \Gamma \times \Gamma$$

$$\nabla e \simeq \Gamma$$

Unphysical Tetrad	$e := \{\ell^a, k^a, m^a, m^{*a}\}$
Directional derivatives	$\nabla = \{D = \ell^a \nabla_a, \Delta = k^a \nabla_a, \delta = m^a \nabla_a, \delta^*\}$
Unphysical connection	$\Gamma = \{\kappa, \tau, \sigma, \rho, \epsilon, \gamma, \beta, \alpha, \pi, \nu, \mu, \lambda\}$
Rescaled Weyl tensor	$\phi = \{\phi_0, \phi_1, \phi_2, \phi_3, \phi_4\}$
Unphysical Trace-free Ricci	$\Phi = \{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{12}, \Phi_{22}\}$

The decoupling of perts of the conf. factor to linear order

If $(\overset{\circ}{\mathcal{M}}, \overset{\circ}{\mathbf{g}})$ is Petrov D then $(\overset{\circ}{\mathcal{M}}, \overset{\circ}{\mathbf{g}}, \overset{\circ}{\Xi})$ is Petrov D, furthermore if the unphysical tetrad aligned to the physical PNDs:

$$\overset{\circ}{\phi}_0 = \overset{\circ}{\phi}_1 = \overset{\circ}{\phi}_3 = \overset{\circ}{\phi}_4 = 0. \quad \overset{\circ}{\kappa} = \overset{\circ}{\sigma} = \overset{\circ}{\nu} = \overset{\circ}{\lambda} = 0.$$

$$\Rightarrow \begin{cases} (\delta^* - 4\alpha + \pi)\phi_0 - (D - 4\rho - 2\epsilon)\phi_1 - 3\kappa\phi_2 = 0, \\ (\Delta - 4\gamma + \mu)\phi_0 - (\delta - 4\tau - 2\beta)\phi_1 - 3\sigma\phi_2 = 0, \\ (D - \rho - \rho^* - 3\epsilon + \epsilon^*)\sigma - (\delta - \tau + \pi^* - \alpha^* - 3\beta)\kappa = \Xi\phi_0 \end{cases}$$

$$\begin{aligned} \Xi\phi_0 &= (\overset{\circ}{\Xi} + \varepsilon\overset{\circ}{\Xi}^{(1)} + \varepsilon^2\overset{\circ}{\Xi}^{(2)} + \dots)(\overset{\circ}{\phi}_0 + \varepsilon\overset{\circ}{\phi}_0^{(1)} + \varepsilon^2\overset{\circ}{\phi}_0^{(2)} + \dots) \\ &= \varepsilon\overset{\circ}{\Xi}\overset{\circ}{\phi}_0^{(1)} + \varepsilon^2(\overset{\circ}{\Xi}^{(1)}\overset{\circ}{\phi}_0^{(1)} + \overset{\circ}{\Xi}\overset{\circ}{\phi}_0^{(2)}) + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon\overset{\circ}{\Xi}\overset{\circ}{\phi}_0^{(1)} + \mathcal{O}(\varepsilon^2) \end{aligned}$$

- CEFEs: decoupling of perturbations of the conformal factor to linear order.
- but for CEFEs to second order there's coupling!
- Contrast: For Conf-Sing-Hyperboloidal there is never a coupling with perts of the conformal factor (conformal factor there is fixed and non dynamical)

CEFE-Teukolsky (G., Panosso Macedo, & Feng 26)

Let $(\overset{\circ}{\mathcal{M}}, \overset{\circ}{\mathbf{g}}, \overset{\circ}{\Xi})$ denote a Petrov-type D solution to the (non-linear) CEFEs. The linearisation of the CEFEs around $(\overset{\circ}{\mathcal{M}}, \overset{\circ}{\mathbf{g}}, \overset{\circ}{\Xi})$ imply the *CEFE-Teukolsky* equation:

$$[(\overset{\circ}{D} - 3\overset{\circ}{\epsilon} + \overset{\circ}{\epsilon}^* - 4\overset{\circ}{\rho} - \overset{\circ}{\rho}^*)(\overset{\circ}{\Delta} - 4\overset{\circ}{\gamma} + \overset{\circ}{\mu}) - (\overset{\circ}{\delta} + \overset{\circ}{\pi}^* - \overset{\circ}{\alpha}^* - 3\overset{\circ}{\beta} - 4\overset{\circ}{\tau})(\overset{\circ}{\delta}^* + \overset{\circ}{\pi} - 4\overset{\circ}{\alpha}) - 3\overset{\circ}{\Xi}\overset{\circ}{\phi}_2]\check{\phi}_0 = 0.$$

$$[(\overset{\circ}{\Delta} + 3\overset{\circ}{\gamma} - \overset{\circ}{\gamma}^* + 4\overset{\circ}{\mu} + \overset{\circ}{\mu}^*)(\overset{\circ}{D} + 4\overset{\circ}{\epsilon} - \overset{\circ}{\rho}) - (\overset{\circ}{\delta}^* - \overset{\circ}{\tau}^* + \overset{\circ}{\beta}^* + 3\overset{\circ}{\alpha} + 4\overset{\circ}{\pi})(\overset{\circ}{\delta}^* - \overset{\circ}{\tau} + 4\overset{\circ}{\beta}) - 3\overset{\circ}{\Xi}\overset{\circ}{\phi}_2]\check{\phi}_4 = 0.$$

Where:

$$\begin{aligned}\overset{\circ}{\phi}_2 &= \overset{\circ}{\Xi}^{-1}\overset{\circ}{\Psi}_2 = \overset{\circ}{\Xi}^{-3}\overset{\circ}{\check{\Psi}}_2, \\ \check{\phi}_0 &= \overset{\circ}{\Xi}^{-1}\overset{\circ}{\check{\Psi}}_0 = \overset{\circ}{\Xi}^{-5-4\omega}b^4e^{2i\vartheta}\overset{\circ}{\check{\Psi}}_0, \\ \check{\phi}_4 &= \overset{\circ}{\Xi}^{-1}\overset{\circ}{\check{\Psi}}_4 = \overset{\circ}{\Xi}^{-1+4\omega}b^{-4}e^{-2i\vartheta}\overset{\circ}{\check{\Psi}}_0,\end{aligned}$$

where (b, ϑ) represents a general spin-boost of the physical null tetrad.

CEFE-Teuk

- Tetrad: adapted to PNDs.
- Coordinates: not specified.

Particular cases (known): reproducing results in the literature

- If background fixed to Kerr and hyperboloidal coords employed it reduces to the hyperboloidal Teukolsky equation for Kerr reported in the hyperboloidal literature (Zenginoglu, Panosso-Macedo, Jaramillo,...) —obtained through an “effective coordinate-based approach”.

Particular cases (new) : Kerr close to i^0 and \mathcal{I}

- If background fixed to Kerr and cylinder-like coordinates introduced by (Hennig & Panosso Macedo 21) \rightsquigarrow (new result) Teuk. eq. for Kerr close to i^0 .
- Technically not F-cylinder at i^0 as coords not based on conf. geodesics.
- Still $i^0 \rightarrow I$ total characteristic and Teuk. eq. reduces to transport equations on I

$$\left((1 - T)(1 + T)\partial_T^2 - 2(1 + \mathfrak{s})\partial_T - \bar{\delta}\bar{\delta} - 2\mathfrak{s} \right) \mathfrak{s}\check{\phi} = 0.$$

where $\mathfrak{s} = 2, 0, -2$ spin-weight. $T = \pm 1$ critical sets where i^0 and \mathcal{I} meet.

Conclusions and Perspectives

Conclusions

- Bridge: linear hyperboloidal \subset linear CEFes
- Decoupling of conformal factor for linear & type D \implies linear hyperboloidal–linear-CEFE agreement
- Gap: second or higher hyperboloidal $\not\subset$ second or higher CEFes.
- Ξ is fixed in hyperboloidal while Ξ satisfies evolution equations in CEFes —magic decoupling only at linear level and type D!

Perspectives

- Metric reconstruction in CEFE framework?
- Beyond linear perturbations?
- Conformal GHP reformulation?
- beyond type D at linear order?
- Further/full development of linear BH pert theory in the CEFes framework