PERIMETER INSTITUTE

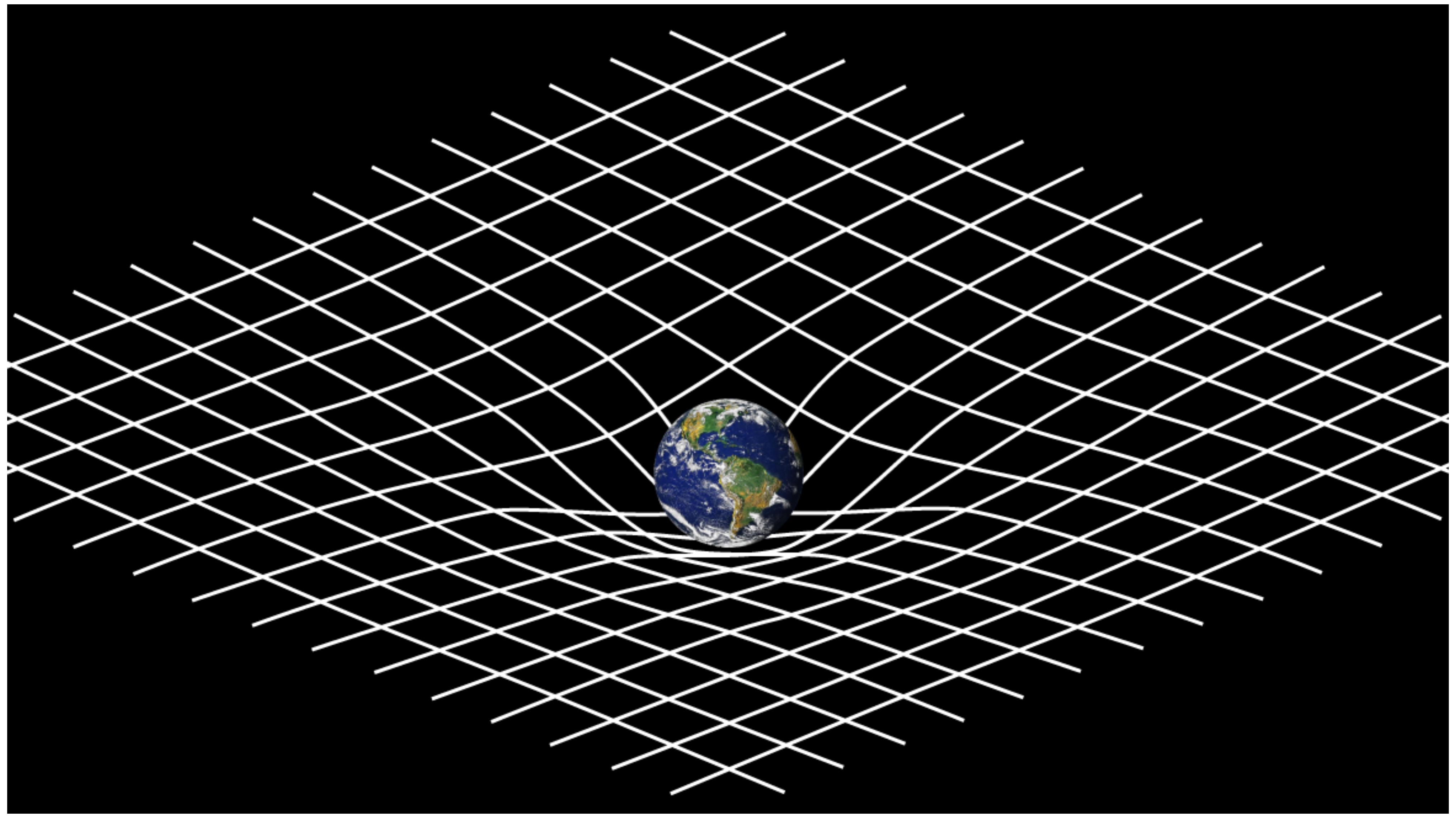
On the conservation of BMS-supertranslation through spatial infinity

Mariem Magdy, 13 Mar 2026

Asymptotics

- Conformal approach [R. Penrose (1960s)]
- Asymptotic flatness?

Definition: A spacetime $(\tilde{M}, \tilde{g})$ is asymptotically simple if it admits a conformal extension (M, g) 'similar' to Minkowski spacetime with $\phi^* g = \Xi^2 \tilde{g}, \phi : \tilde{M} \to M$, so that M has a null boundary $\mathscr{I} \equiv \partial M$ on which $\Xi = 0$ and $d\Xi \neq 0$.

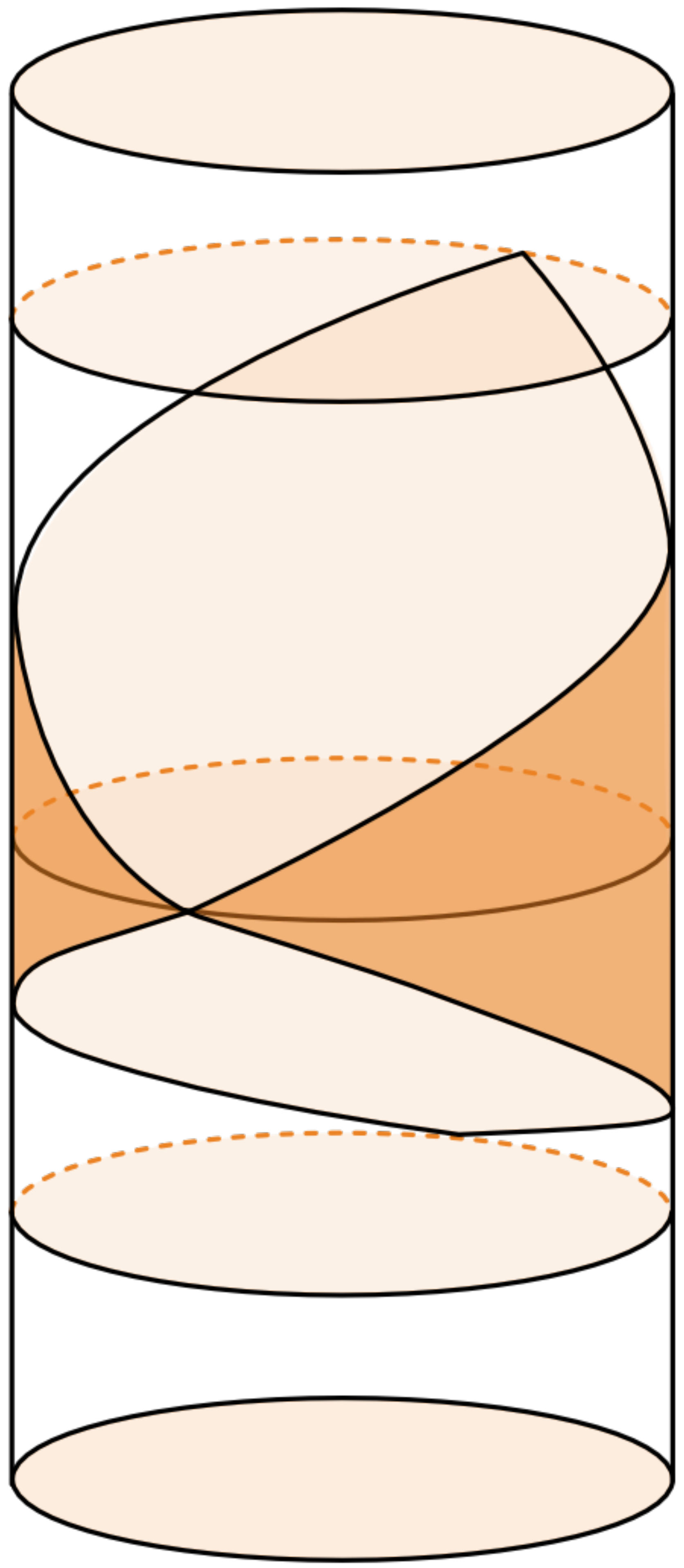


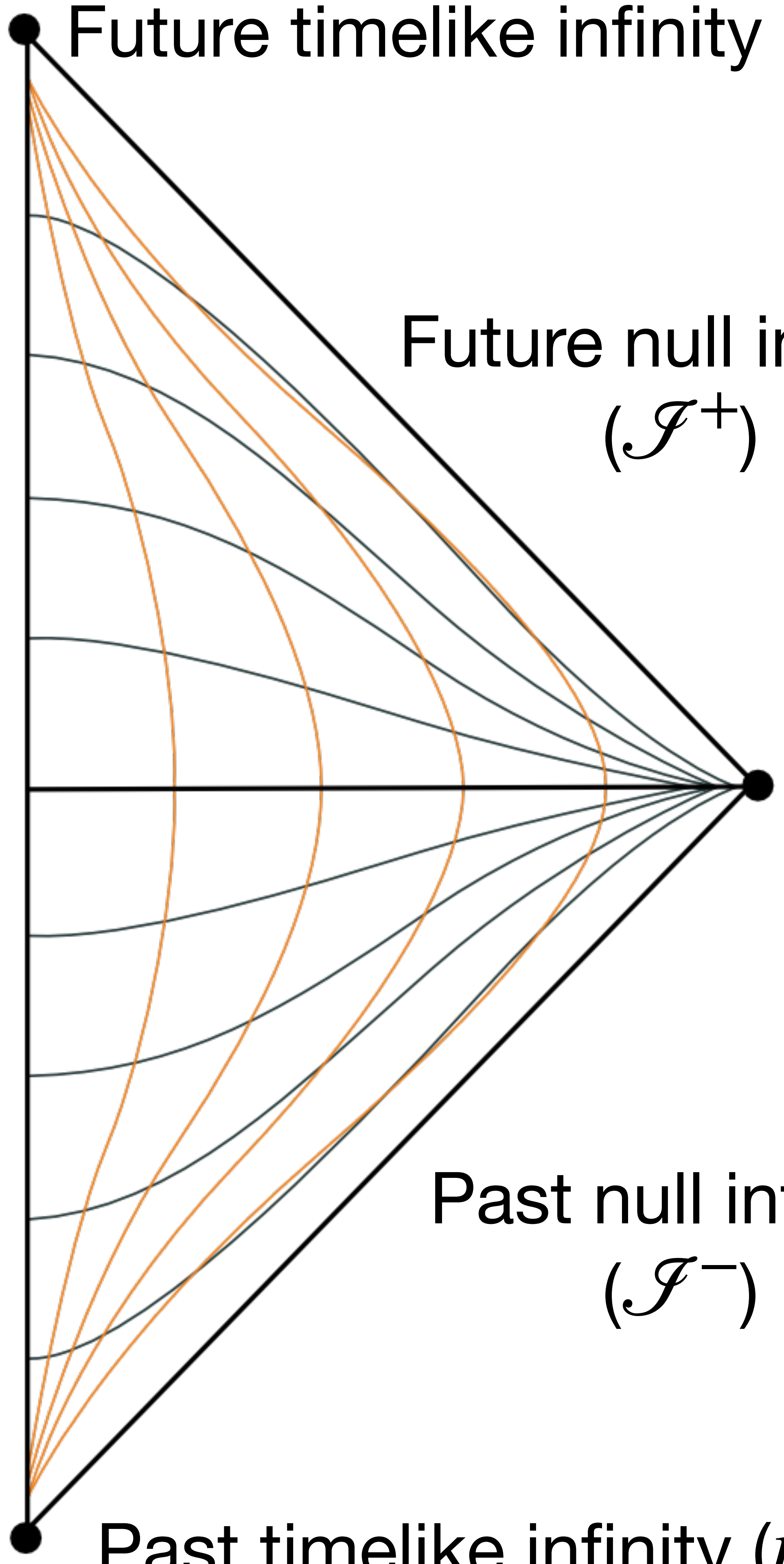
Asymptotics around an isolated object?

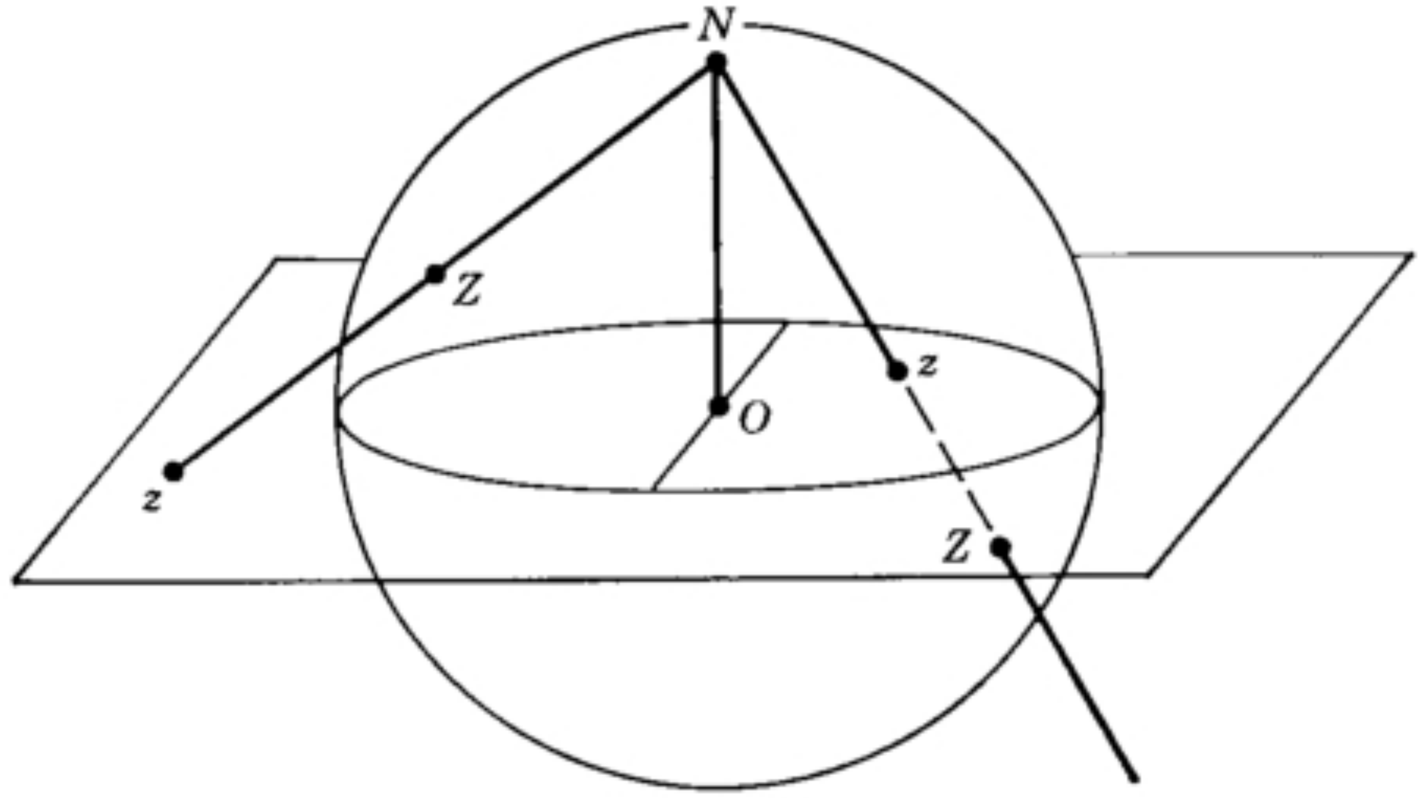
Credit: NASA

Conformal diagram: Minkowski

Embedding of Minkowski $\mathbb{R}^{1,3}$ spacetime into $\mathbb{R} \times \mathbb{S}^3$







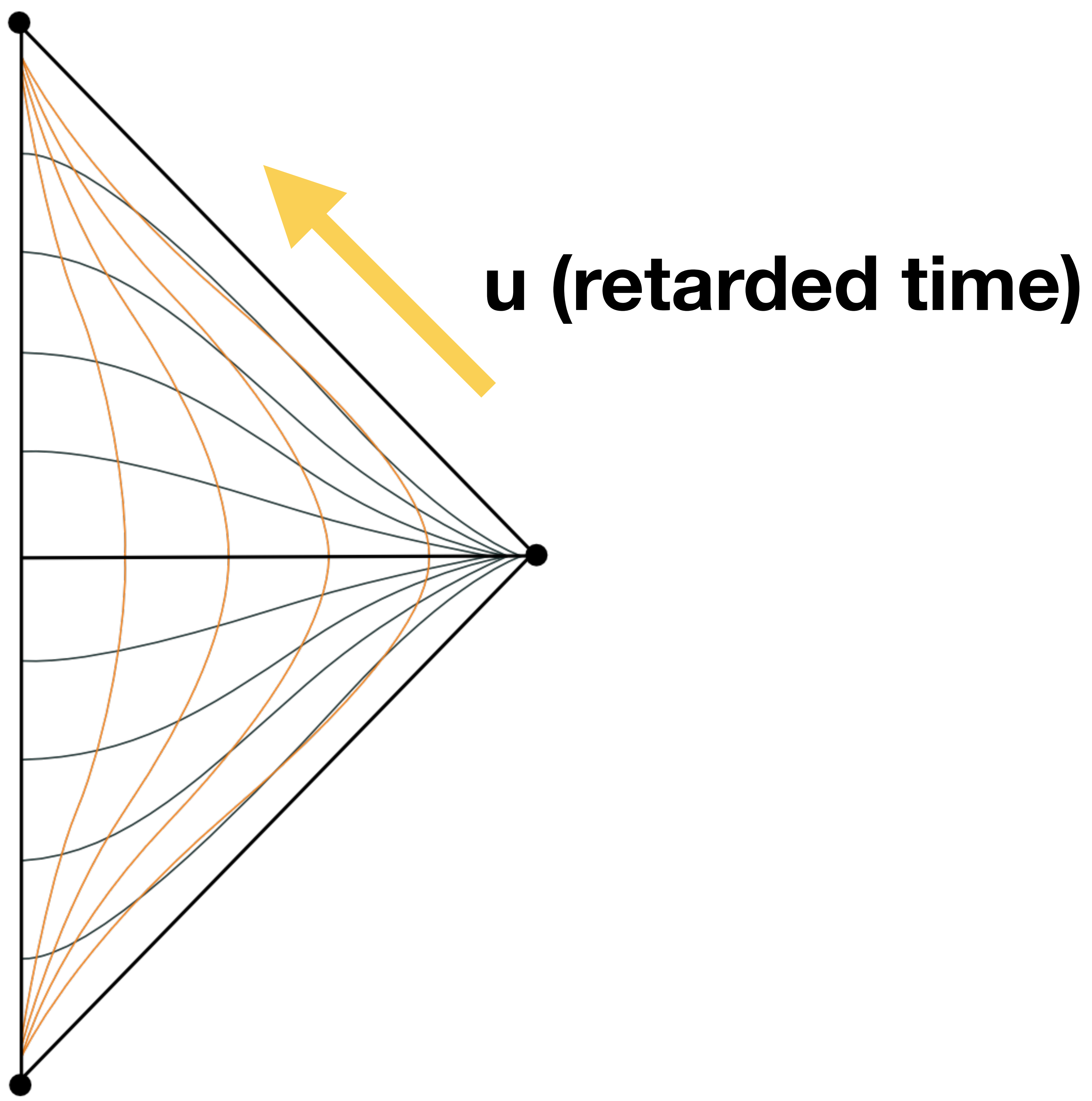
Conformal mapping between $\mathbb{R}^2$ and $\mathbb{S}^2$

Other notions of asymptotic flatness due to Ashtekar, Romano and Hansen (70's and 90's)

More on conformal flatness

Asymptotically flat metric in Bondi coordinates

$$g = r^2\gamma_{ij}dx^i dx^j + rC_{ij}dx^i dx^j + (-du^2 - 2dudr + D^j C_{ij}dudx^i) + \frac{2M}{r}du^2 + \frac{1}{r}(2N_i + \frac{1}{4}C_{ij}D_k C^{jk})dudx^i \ldots$$



More on conformal flatness

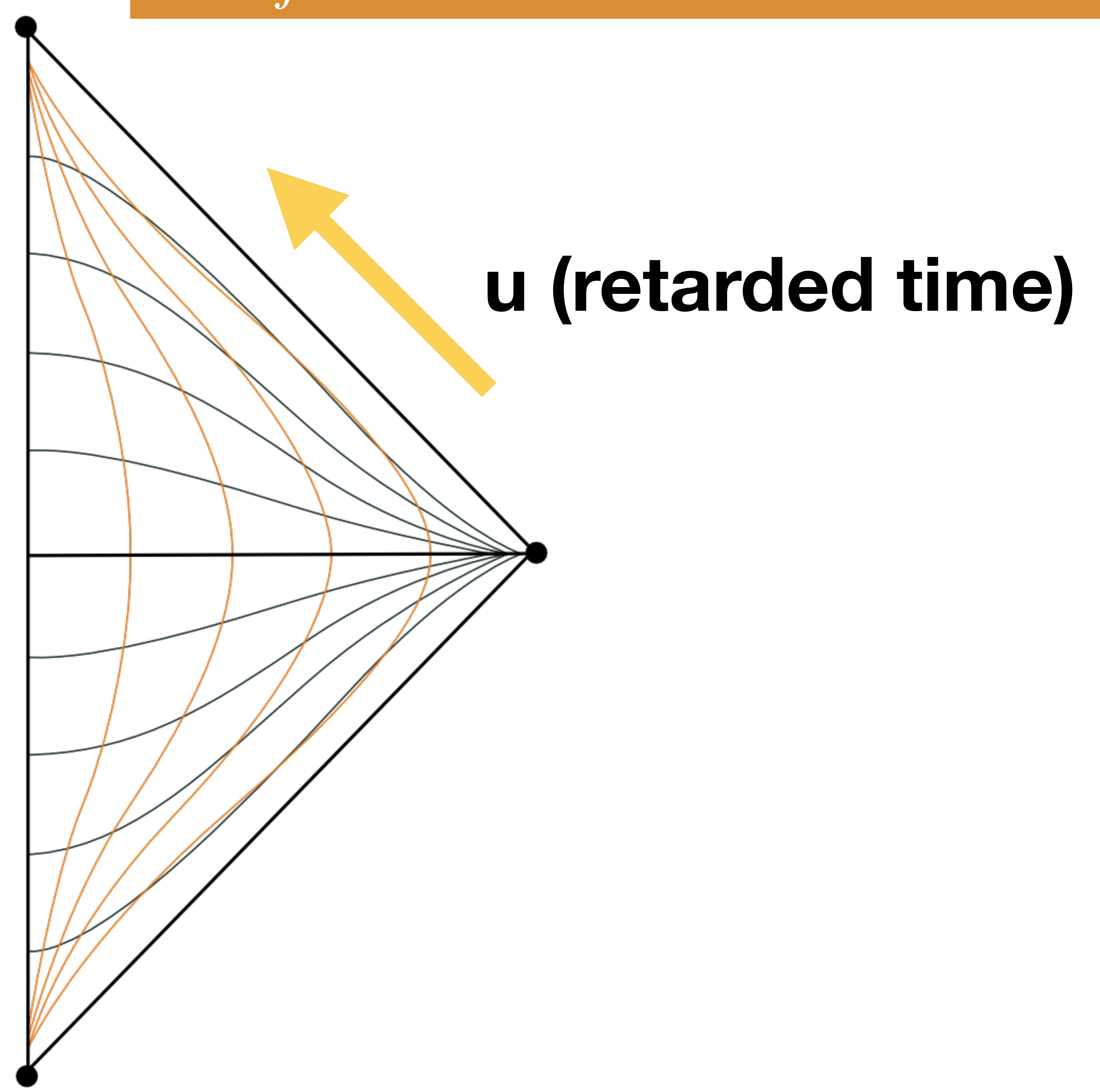
Asymptotically flat metric in Bondi coordinates

$$g = r^2\gamma_{ij}dx^i dx^j + rC_{ij}dx^i dx^j + (-du^2 - 2dudr + D^jC_{ij}dudx^i) + \frac{2M}{r}du^2 + \frac{1}{r}(2N_i + \frac{1}{4}C_{ij}D_kC^{jk})dudx^i \ldots$$

$C_{ij}(u, x^k)$: Asymptotic Shear

$M(u, x^i)$: Bondi mass aspect

$N_i(u, x^i)$: angular mom. aspect



What sort of transformations preserve this metric expansion?

Find $\xi = \xi^u\partial_u + \xi^r\partial_r + \xi^i\partial_i$ that preserves leading order term and the conditions $g_{rr} = 0,\ g_{ri} = 0,\ \det(g_{ij}) = r^2\det(\gamma_{ij})$

More on conformal flatness

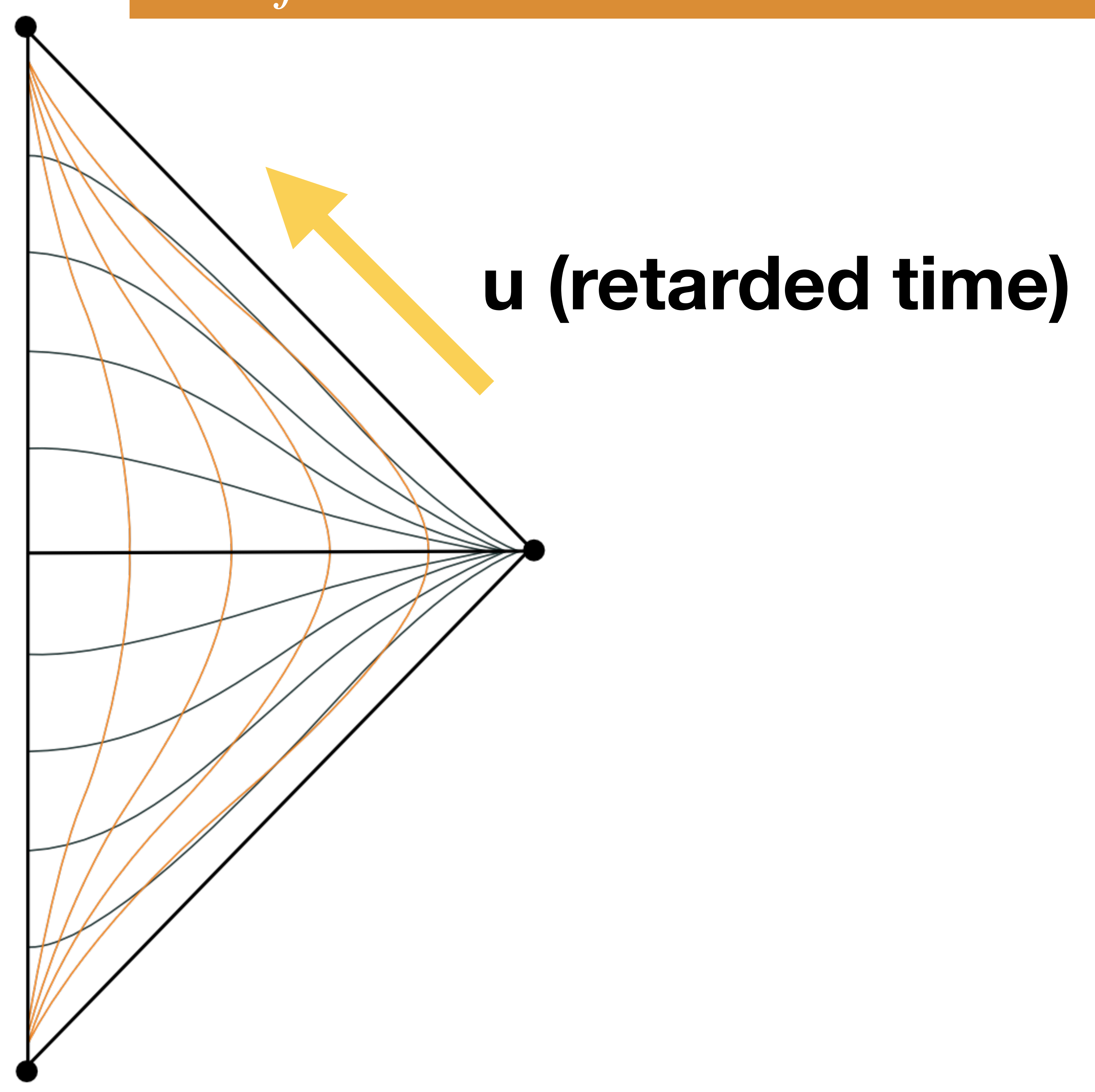
Asymptotically flat metric in Bondi coordinates

$$g = r^2\gamma_{ij}dx^idx^j + rC_{ij}dx^idx^j + (-du^2 - 2dudr + D^jC_{ij}dudx^i) + \frac{2M}{r}du^2 + \frac{1}{r}(2N_i + \frac{1}{4}C_{ij}D_kC^{jk})dudx^i \ldots$$

$C_{ij}(u, x^k)$: Asymptotic Shear

$M(u, x^i)$: Bondi mass aspect

$N_i(u, x^i)$: angular mom. aspect



What sort of transformations preserve this metric expansion?

On $\mathscr{I}^+$, the infinitesimal generator ξ can be written as

$$\xi = T\partial_u + \frac{u}{2}D_iY^i\partial_u + Y^i\partial_i$$

$T(x^i)$: supertranslation parameter

$Y^i(x^j)$: Lorentz parameter

More on conformal flatness

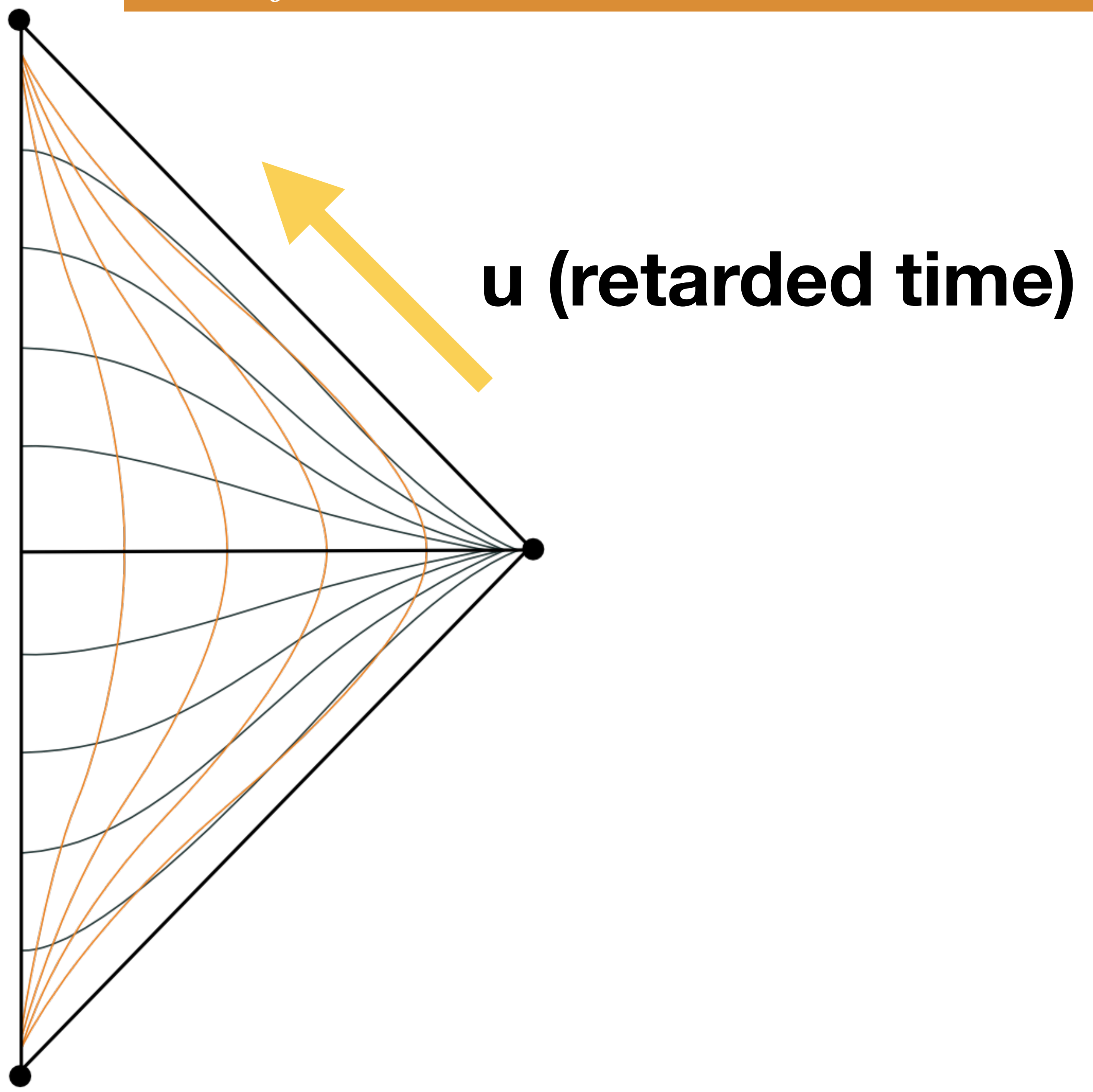
Asymptotically flat metric in Bondi coordinates

$$g = r^2\gamma_{ij}dx^i dx^j + rC_{ij}dx^i dx^j + (-du^2 - 2dudr + D^j C_{ij} dudx^i) + \frac{2M}{r}du^2 + \frac{1}{r}(2N_i + \frac{1}{4}C_{ij}D_k C^{jk})dudx^i \ldots$$

$C_{ij}(u, x^k)$: Asymptotic Shear

$M(u, x^i)$: Bondi mass aspect

$N_i(u, x^i)$: angular mom. aspect



What sort of transformations preserve this metric expansion?

A: The BMS group preserve this expansion and has the structure

$$\mathfrak{bms}_4 \simeq C^\infty(\mathbb{S}^2) \ltimes \mathfrak{so}(3,1)$$

Definition of BMS asymptotic charges

- The Bondi 4-momentum on $\mathscr{C} \subset \mathscr{I}^+$

$$P_{l,m} = \frac{1}{4\pi} \oint_{\mathscr{C}} M \, Y_{l,m} \, \epsilon_2, \qquad l = 0,1, \quad -l \leq m \leq l$$

Represents total energy-momentum of the spacetime measured at null infinity

- Definition of BMS charges e.g. [Wald, Zoupas (2000)] [Barnich, Troessaert (2011)][Grant, Prabhu, Shehzad (2022)]

$$Q[T, Y; \mathscr{C}] = -\frac{1}{8\pi} \oint_{\mathscr{C}} \left[-2M \, \beta + Y^i \left(3N_i - \frac{1}{32} D_i (C_{jk} C^{jk}) - \frac{1}{2} C_i{}^j D^k C_{jk} \right) \right] \epsilon_2,$$

$$\beta := T + \frac{u}{2} D_i Y^i$$

Definition of BMS asymptotic charges

- The Bondi 4-momentum on $\mathscr{C} \subset \mathscr{I}^+$

$$P_{l,m} = \frac{1}{4\pi} \oint_{\mathscr{C}} M\, Y_{l,m}\, \epsilon_2, \qquad l = 0,1, \quad -l \leq m \leq l$$

Represents total energy-momentum of the spacetime measured at null infinity

- Definition of BMS charges e.g. [Wald, Zoupas (2000)] [Barnich, Troessaert (2011)][Grant, Prabhu, Shehzad (2022)]

$$Q[T, Y; \mathscr{C}] = -\frac{1}{8\pi} \oint_{\mathscr{C}} \left[-2M\, T - uM\, D_i Y^i + Y^i \left(3N_i - \frac{1}{32} D_i (C_{jk} C^{jk}) - \frac{1}{2} C_i{}^j D^k C_{jk} \right) \right] \epsilon_2,$$

Supertranslation charges

Definition of BMS asymptotic charges

- The Bondi 4-momentum on $\mathscr{C} \subset \mathscr{I}^+$

$$P_{l,m} = \frac{1}{4\pi}\oint_{\mathscr{C}} M\, Y_{l,m}\, \epsilon_2, \qquad l = 0,1, \quad -l \leq m \leq l$$

Represents total energy-momentum of the spacetime measured at null infinity

- Definition of BMS charges e.g. [Wald, Zoupas (2000)] [Barnich, Troessaert (2011)][Grant, Prabhu, Shehzad (2022)]

$$Q[T, Y; \mathscr{C}] = -\frac{1}{8\pi}\oint_{\mathscr{C}} \left[-2M\, T - uM\, D_i Y^i + Y^i\left(3N_i - \frac{1}{32}D_i(C_{jk}C^{jk}) - \frac{1}{2}C_i{}^j D^k C_{jk}\right)\right]\epsilon_2,$$

Lorentz charges

The BMS-supertranslation charges

$$Q[T;\mathscr{C}] = \frac{1}{4\pi}\oint_{\mathscr{C}} M\, T\, \epsilon_2,$$

To the NP-gauge

$$Q[T;\mathscr{C}] = \frac{1}{4\pi}\oint_{\mathscr{C}} \left(\mathscr{P} + \frac{1}{2}\sigma^{ab}N_{ab}\right) T\, \epsilon_2,$$

N_{ab}: the Bondi news

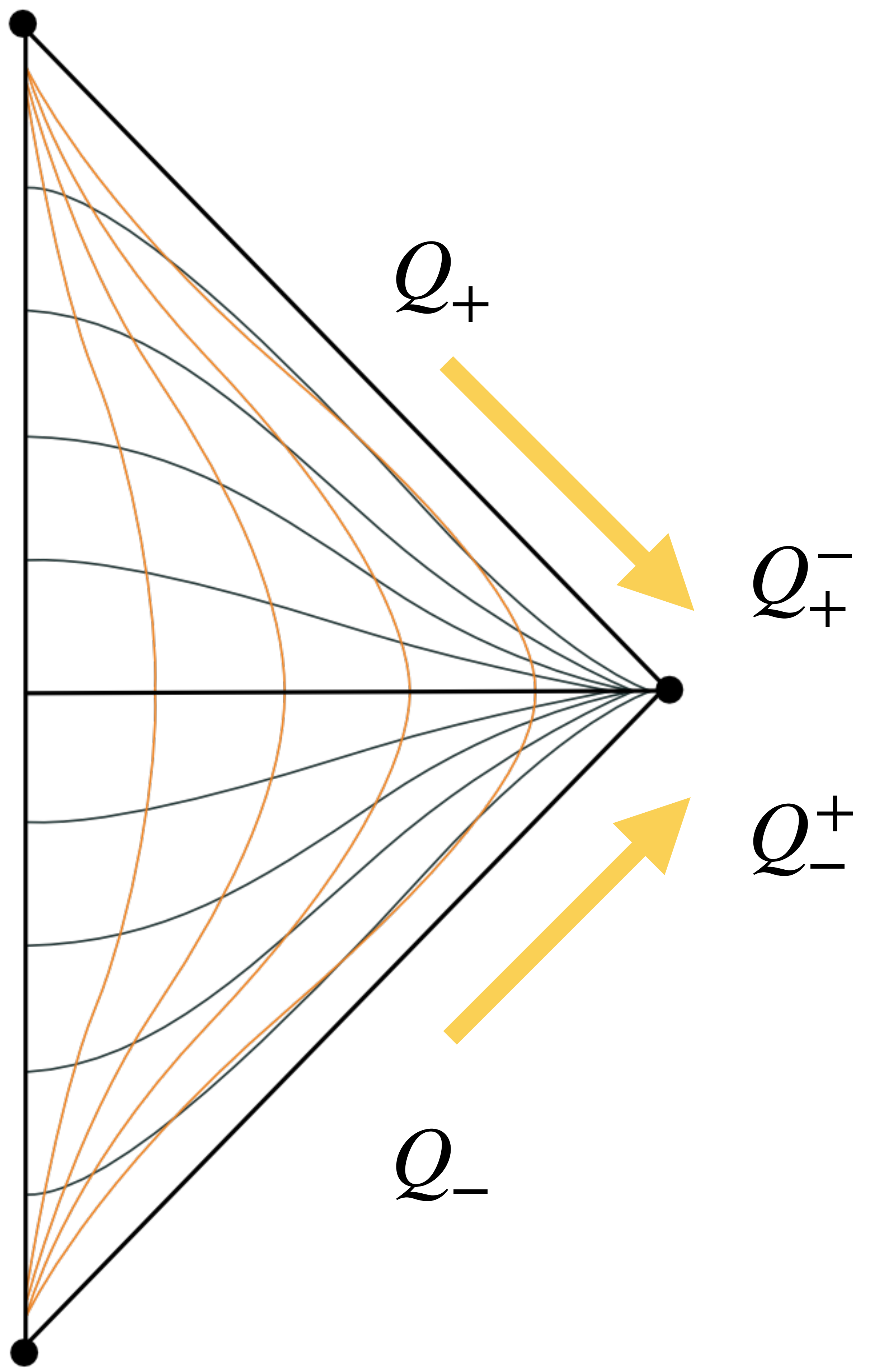
σ^{ab}:Shear

- The BMS-supertranslation charges depends on:

$\mathscr{P}$: Coulomb part of the Weyl tensor $\Psi_2 := -C_{abcd}l^a m^b \bar{m}^c n^d$

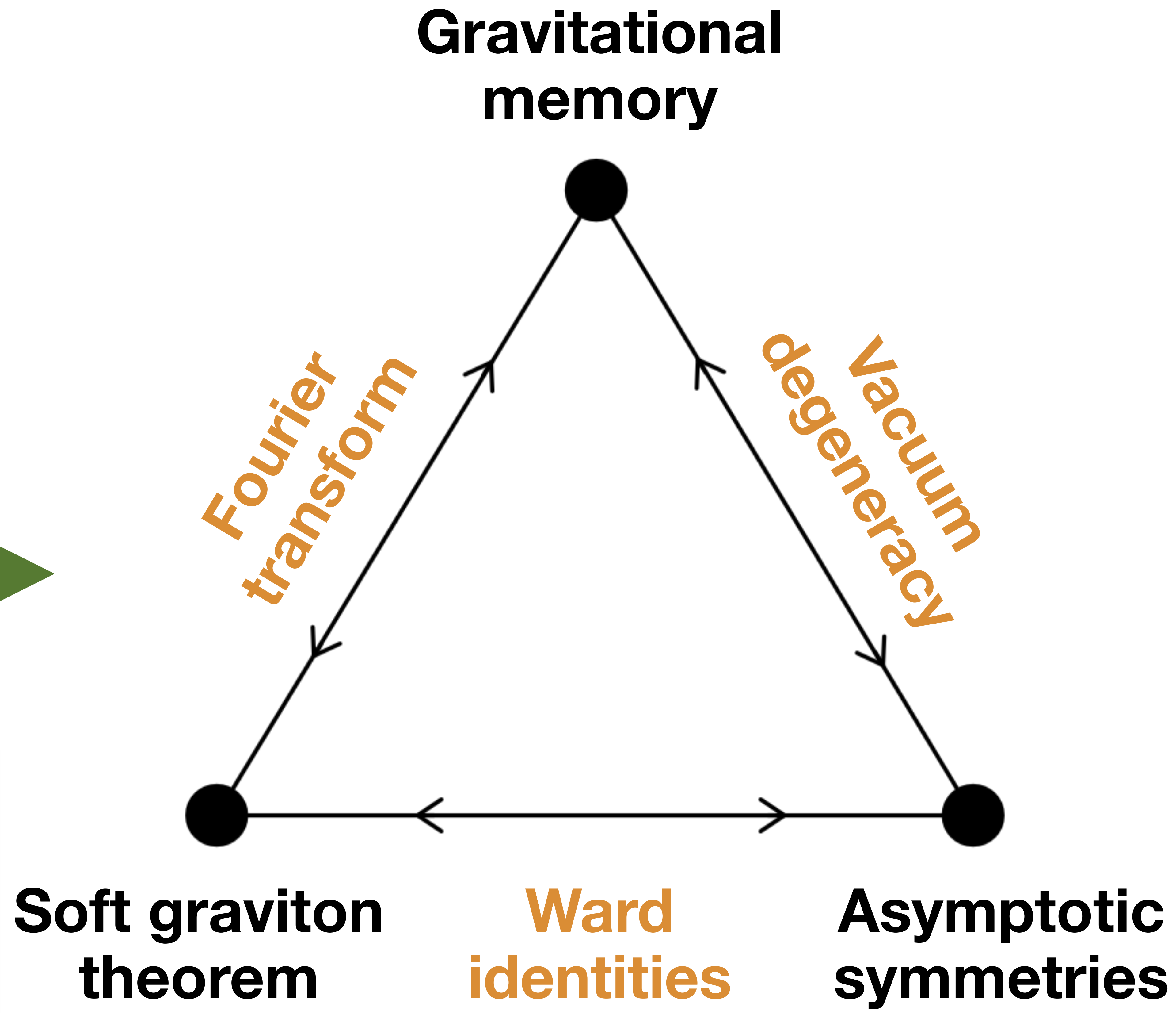
The contraction $\sigma^{ab}N_{ab}$ which can be written at $\mathscr{I}^+$ as
$\sigma^{ab}N_{ab} = 2\delta(\sigma\bar{\sigma}) - \sigma\bar{\sigma}(3\mu + 3\bar{\mu} + \gamma + \bar{\gamma})$

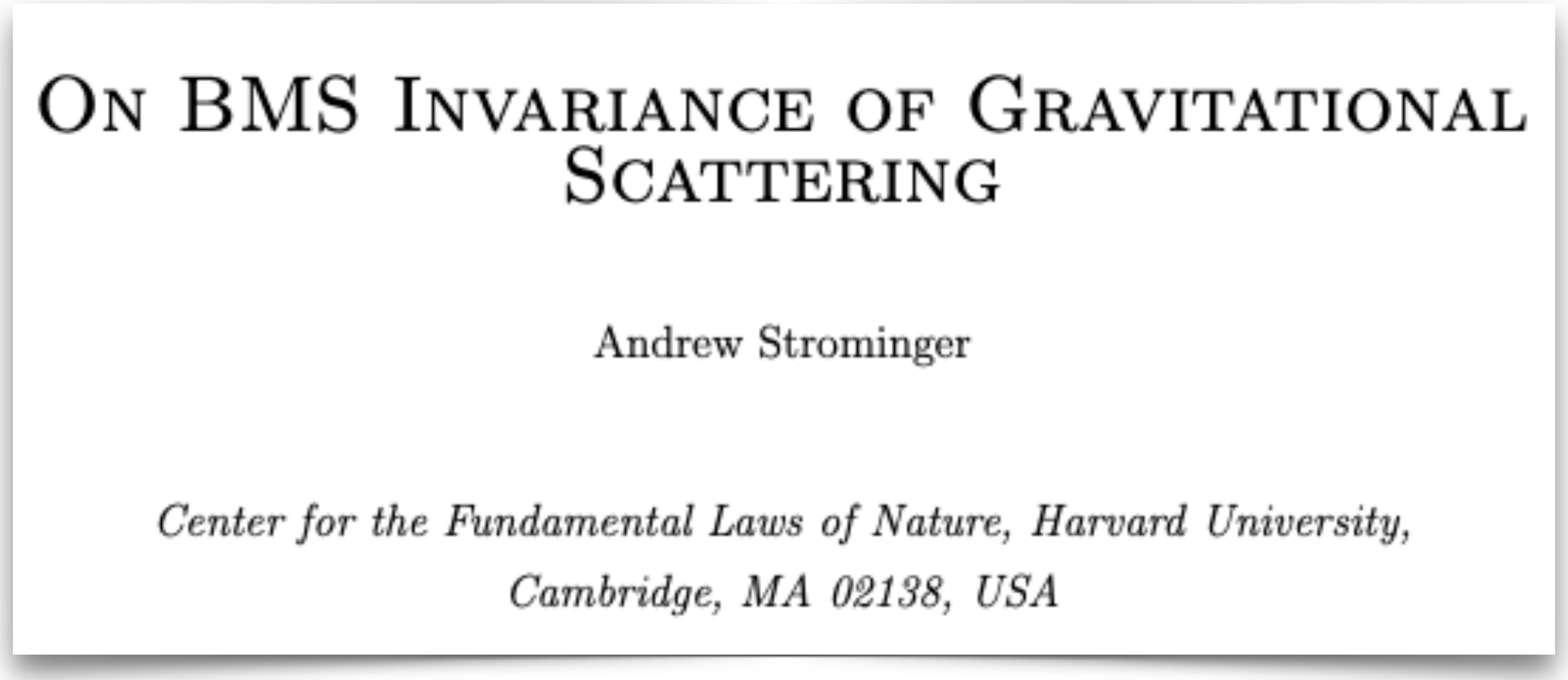
The conservation of BMS-supertranslation



Key question: how the charges defined at past null infinity are related to the charges defined at future null infinity?

If $Q_+^- \propto Q_-^+$





On BMS Invariance of Gravitational Scattering

Andrew Strominger

Center for the Fundamental Laws of Nature, Harvard University, Cambridge, MA 02138, USA

Some matching results

Spin-1 and Spin-2

- [Campiglia, Eyheralde (2017)].
- [Henneaux, Troessaert (2018)].
- [Prabhu (2018)].
- [Troessaert (2018)].
- *[Magdy, Valiente-Kroon (2022)]*

GR

- [Prabhu (2019)].
- *[Magdy, Prabhu, Valiente-Kroon (2024)]*

Spin-0

- [Fuentealba, Henneaux (2024)]
- *[Gasperín, Magdy, Mena (2024)]*
- [Briceño, González, Henneaux, Pérez (2025)]

Friedrich's approach to spatial infinity

Friedrich's approach → Blow-up of spatial infinity — To resolve singular behaviour at i^0 due to point compactification

Friedrich's approach → Friedrich's conformal field eqs. — To introduce well-defined equations at the conformal boundary

- Take $(\tilde{M}, \tilde{\mathbf{g}})$ satisfying $\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} = 0$ and $g = \Xi^2 \tilde{g}$

Observation: The conformal transformation of the Ricci tensor $\tilde{R}_{ab} \to R_{ab}$ implies a singular equation for g_{ab} at $\partial\mathscr{U} \equiv \{\Xi = 0\}$

Friedrich's approach to spatial infinity

Friedrich's approach

→ **Blow-up of spatial infinity** — To resolve singular behaviour at i^0 due to point compactification

→ **Friedrich's conformal field eqs.** — To introduce well-defined equations at the conformal boundary

- Use Friedrich's Extended conformal field equations (ECFE)

Background

$$[\mathbf{e}_b, \mathbf{e}_a] - (\hat{\Gamma}_a{}^c{}_b - \hat{\Gamma}_b{}^c{}_a)\mathbf{e}_c = 0$$
$$\hat{P}^c{}_{dab} - \hat{\rho}^c{}_{dab} = 0$$
$$\hat{\nabla}_c \hat{L}_{db} - \hat{\nabla}_d \hat{L}_{cb} - d_a d^a{}_{bcd} = 0$$

Gauge fixing

$$d_a - \Xi f_a - \hat{\nabla}_a \Xi = 0$$
$$\hat{L}_{ab} - \hat{\nabla}_a \beta_b - \frac{1}{2} S_{ab}{}^{cd} \beta_c \beta_d = 0$$
$$\hat{L}_{[ab]} - \hat{\nabla}_{[a} f_{b]} = 0$$

Gravitational field

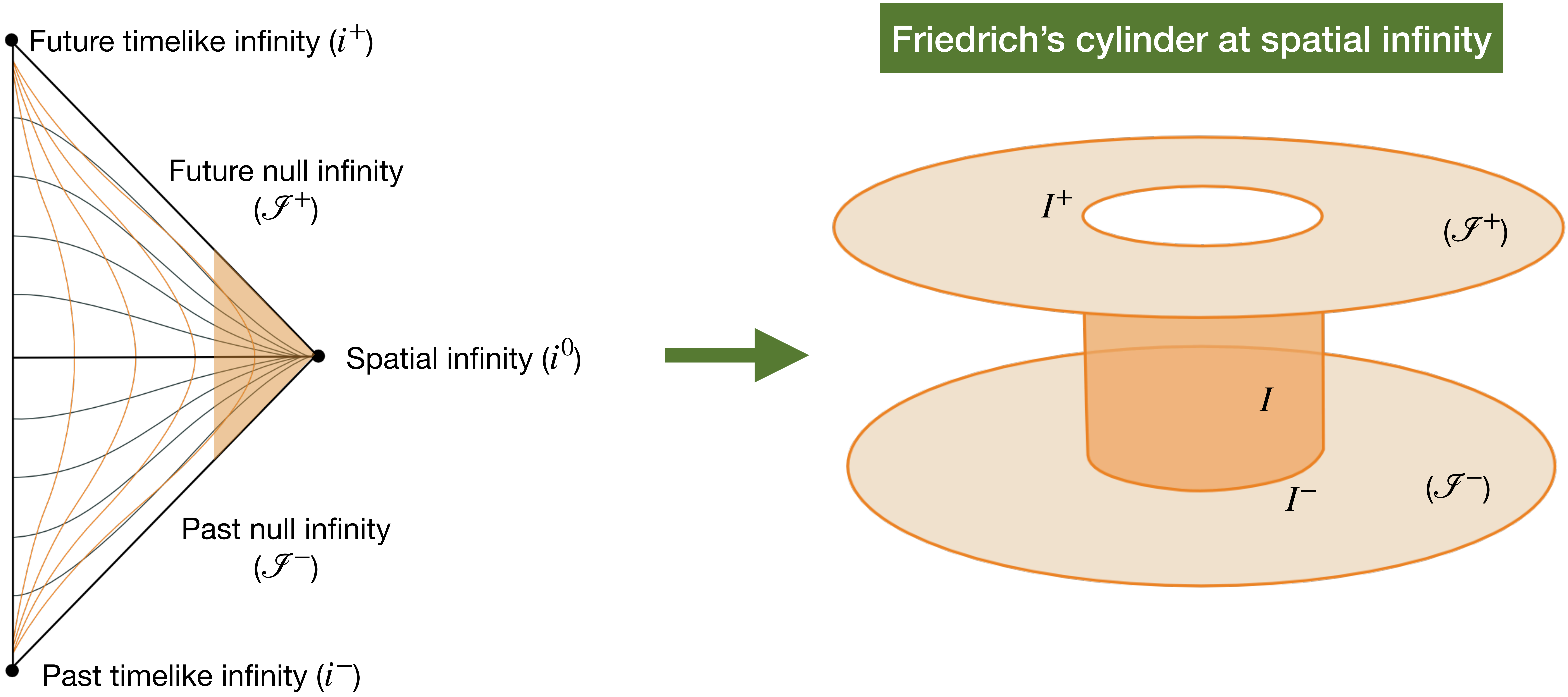
$$\hat{\nabla}_a d^a{}_{bcd} - f_a d^a{}_{bcd} = 0$$

$\mathbf{e}_a$: orthonormal frame

$\hat{\nabla}$: Weyl connection $\hat{\nabla}_a g_{bc} = -2 f_a g_{bc}$

[Friedrich (1981) (1983), Valiente-Kroon (2016)]

Friedrich’s blow up on Minkowski



Friedrich's blow up on Minkowski

- Start with $\tilde{\eta} = d\tilde{t} \otimes d\tilde{t} - d\tilde{r} \otimes d\tilde{r} - \tilde{r}^2\sigma$
- Introduce coordinate transformation $\tau = \frac{\tilde{t}}{\tilde{r}}, \quad \rho = -\frac{\tilde{r}}{\tilde{t}^2 - \tilde{r}^2},$
- And a conformal transformation $(\mathbb{R}^{1,3}, \tilde{\eta}) \to (\mathscr{M}, \eta)$: $\eta = \Theta^2\tilde{\eta}, \quad$ with $\Theta = \rho(1-\tau^2)$

Friedrich's conformal metric

$$\eta = d\tau \otimes d\tau - \frac{(1-\tau^2)}{\rho^2} d\rho \otimes d\rho + \frac{\tau}{\rho}(d\tau \otimes d\rho + d\rho \otimes d\tau) - \sigma$$

- Conformal boundary defined by $\Theta = 0 \to \tau = \pm 1$ or $\rho = 0$

$$\mathscr{I}^{\pm} \equiv \{p \in \mathscr{M} \mid \tau(p) = \pm 1\},$$ Past/future null infinity

$$I^{\pm} \equiv \{p \in \mathscr{M} \mid \tau(p) = \pm 1, \rho(p) = 0\},$$ Critical sets of null infinity

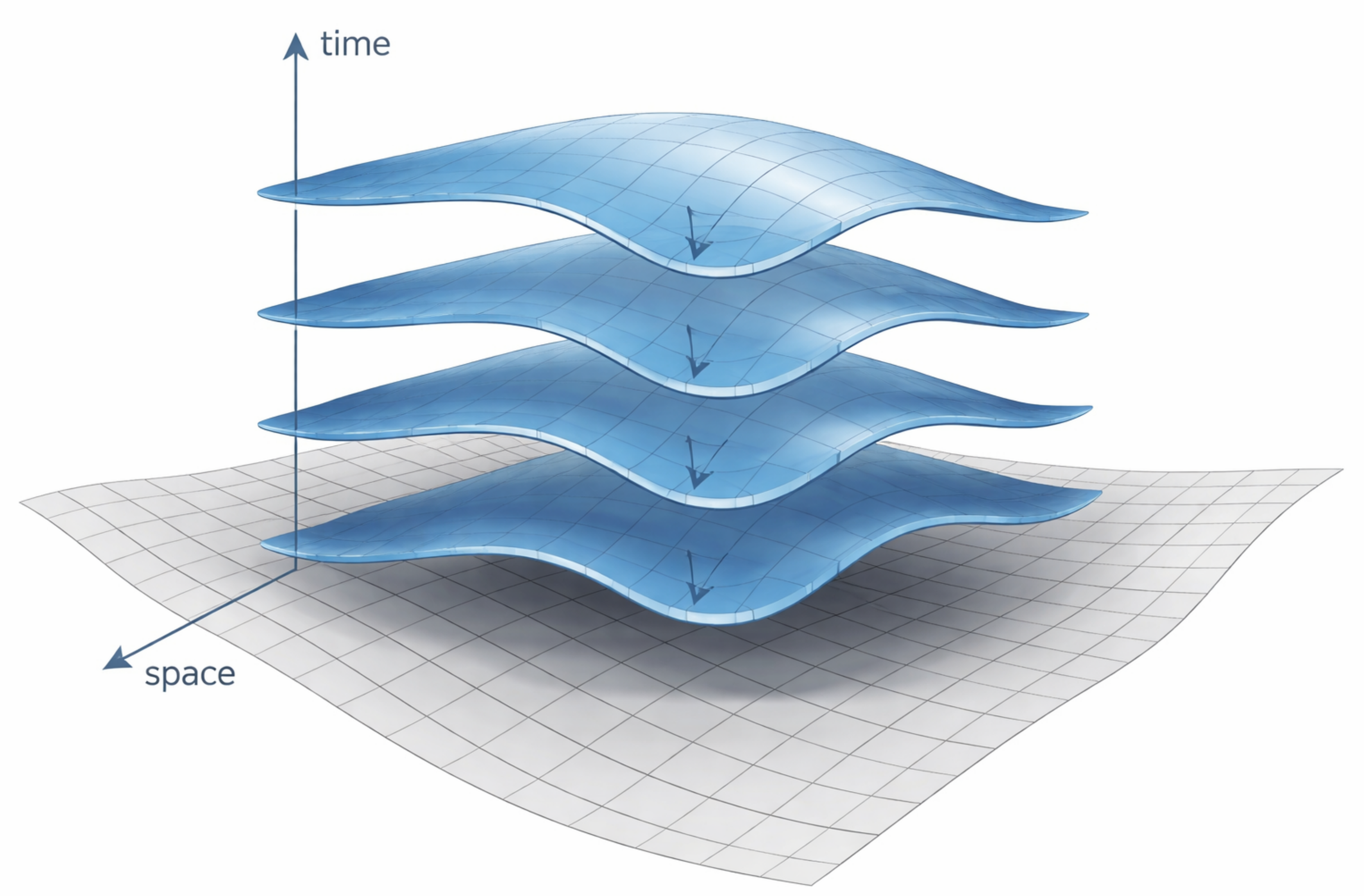
$$I \equiv \{p \in \mathscr{M} \mid \rho(p) = 0\},$$ Cylinder at spatial infinity

Friedrich's blow up on asymptotically flat spacetimes

- Friedrich's formulation is an IVP formulation → 3+1 decomposition

- Asymptotically flat spacetimes develops from 'asymptotically Euclidean data'

Definition (informal) [Geroch (1972)]: A 3-dimensional Riemannian manifold $(\tilde{\mathcal{S}}, \tilde{h})$ is asymptotically Euclidean and regular if it's geometry approaches that of flat Euclidean space far away from sources. In the conformal picture, $\varepsilon_k \subset \tilde{\mathcal{S}} \to i_k$ (spatial infinity) at which the rescaled geometry satisfies specific regularity conditions.



Foliation of spacetime with spacelike slices $\tilde{\mathcal{S}}$

Friedrich's blow up on asymptotically flat spacetimes

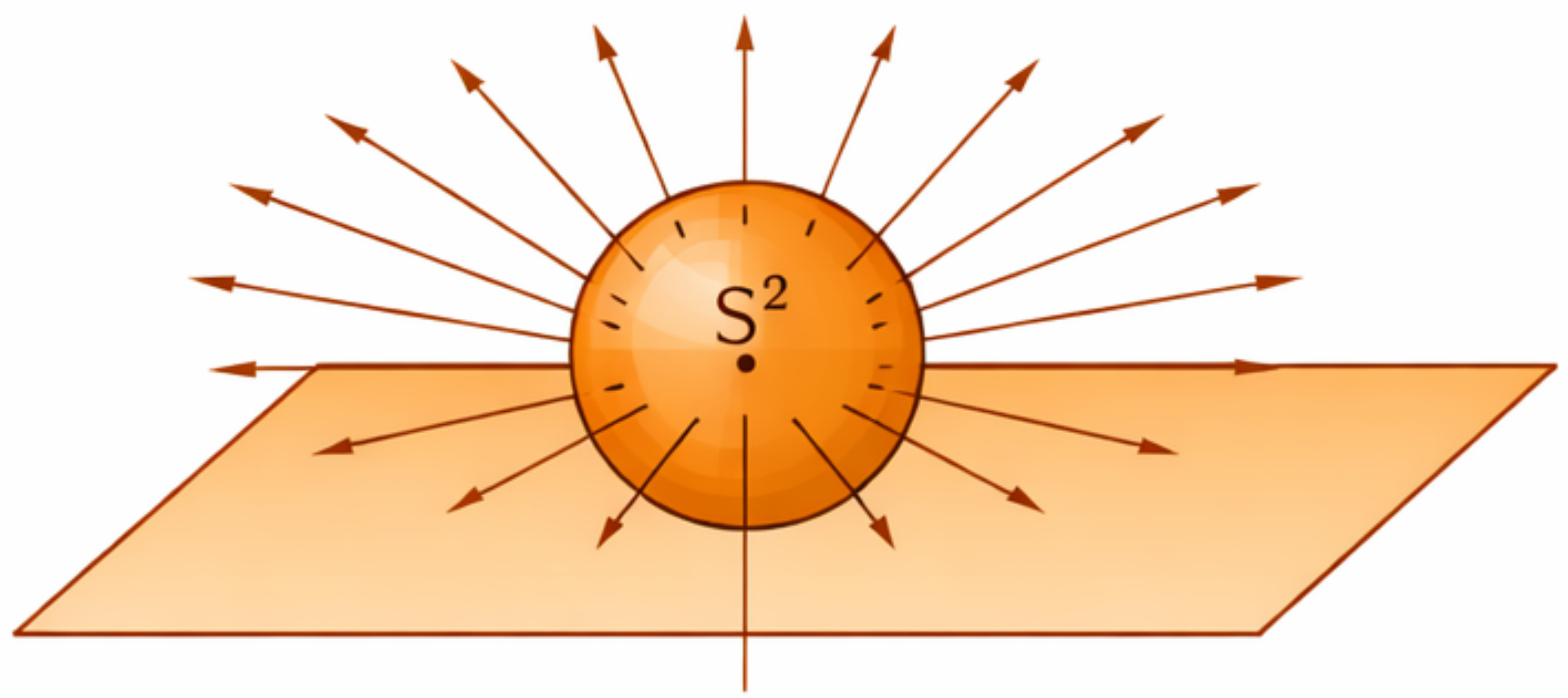
Blow-up of spatial infinity

- Assume $(\tilde{S}, \tilde{h})$ with one asymptotic end $\varepsilon \to i$

- Replace i with

Classical blow-up: space of direction at i

Friedrich's blow-up: a subset of the bundle of normalised spin frames $SU(\mathcal{S})$ with structure group $SU(2,\mathbb{C})$



Blow-up on initial hypersurface

(Friedrich's blow-up): blow up of $i \cong \mathbb{S}^3$ rather than the standard blow-up where $i \cong \mathbb{S}^2$

Extra dimension reflects the freedom in the choice of the phase parameter of the spin frame

Friedrich's blow up on asymptotically flat spacetimes

Ingredients for Friedrich's blow-up

$\{\epsilon_{\mathbf{A}}\}$: normalised spin frame at i satisfying $\epsilon_{\mathbf{A}}{}^{A}\epsilon^{\mathbf{B}}{}_{A} = \delta_{\mathbf{A}}{}^{\mathbf{B}}$

$\tau^{AA'}$: timelike vector field satisfying $\tau_{AA'}\tau^{AA'} = 2$

$\mathbf{t} \in SU(2,\mathbb{C})$

$\rho, \tau \in \mathbb{R}$

Recipe for Friedrich's blow-up

- Take $\mathbf{t} \in SU(2,\mathbb{C})$, then $\epsilon_{\mathbf{A}} \to \epsilon_{\mathbf{A}}(\mathbf{t}) = t_{\mathbf{A}}{}^{\mathbf{B}}\epsilon_{\mathbf{B}}$
- Take open ball $B_a(i)$: Extend $\epsilon_{\mathbf{A}}(\mathbf{t})$ to $B_a(i)$ by parallel propagation along geodesics $\gamma_i(\rho)$. So $\epsilon_{\mathbf{A}}(\mathbf{t}) \to \epsilon_{\mathbf{A}}(\rho, \mathbf{t})$
- Extend $\epsilon_{\mathbf{A}}(\rho, \mathbf{t})$ off initial hypersurface: by parallel propagation along conformal geodesics $(x(\tau), \beta_a(\tau))$. So $\epsilon_{\mathbf{A}}(\rho, \mathbf{t}) \to \epsilon_{\mathbf{A}}(\tau, \rho, \mathbf{t})$

Motivation: Transformations $\mathbf{t} \in SL(2,\mathbb{C})$ preserve $[\![\xi, \lambda]\!] = \varepsilon_{AB}\,\xi^A\lambda^B$ whereas $\mathbf{t} \in SU(2,\mathbb{C}) \subset SL(2,\mathbb{C})$ also preserves Hermitian inner product $\langle \xi, \lambda \rangle = \tau_{AA'}\,\bar{\xi}^{A'}\lambda^A$

Friedrich's blow up on asymptotically flat spacetimes

- A choice of $(\tau, \rho, \mathbf{t})$ allows us to propagate spin frames into the spacetime

$$\mathscr{M}_{a,\kappa} = \left\{ (\tau, \rho, \mathbf{t}) \in \mathbb{R} \times \mathbb{R} \times SU(2,\mathbb{C}) \,|\, 0 \leq \rho < a, -\frac{\omega}{\kappa} \leq \tau \leq \frac{\omega}{\kappa} \right\},$$

- Subsets of conformal boundary

$$\mathscr{I}_a^{\pm} = \{(\tau, \rho, \mathbf{t}) \in \mathscr{M}_{a,\kappa} \,|\, 0 < \rho < a, \tau = \pm \frac{\omega}{\kappa}\},$$

$$I^{\pm} = \{(\tau, \rho, \mathbf{t}) \in \mathscr{M}_{a,\kappa} \,|\, \rho = 0, \tau = \pm 1\},$$

$$I = \{(\tau, \rho, \mathbf{t}) \in \mathscr{M}_{a,\kappa} \,|\, \rho = 0, -1 < \tau < 1\},$$

κ: arbitrary smooth function satisfying $\kappa = \mathscr{O}(\rho)$

ω: smooth function satisfying $\omega/\kappa \to 1$ as $\rho \to 0$

Matching of BMS-supertranslation

- We consider the matching of

$$Q[T;\mathscr{C}] = \frac{1}{4\pi}\oint_{\mathscr{C}} \left(\mathscr{P} + \frac{1}{2}\sigma^{ab}N_{ab}\right) T\, \epsilon_2,$$

- Start from asymptotically Euclidean and regular initial data [L. Huang (2010)]

$$h_{\alpha\beta} = -\left(1+\frac{A}{r}\right)\delta_{\alpha\beta} - \frac{\xi}{r}\left(\frac{x_\alpha x_\beta}{r^2} - \frac{1}{2}\delta_{\alpha\beta}\right) + O_2(r^{-2})$$

$$\pi_{\alpha\beta} = \frac{\zeta}{r^2}\frac{x_\alpha x_\beta}{r^2} + \frac{1}{r^3}\left(-B_\alpha x_\beta - B_\beta x_\alpha + (B^\gamma x_\gamma)\delta_{\alpha\beta}\right) + O_1(r^{-3})$$

Matching of BMS-supertranslation

- We consider the matching of

$$Q[T;\mathscr{C}] = \frac{1}{4\pi}\oint_{\mathscr{C}} (\mathscr{P} + \frac{1}{2}\sigma^{ab}N_{ab})\ T\ \epsilon_2,$$

- Start from asymptotically Euclidean and regular initial data [L. Huang (2010)]

$$h_{\alpha\beta} = -\left(1 + \frac{A}{r}\right)\delta_{\alpha\beta} - \frac{\xi}{r}\left(\frac{x_\alpha x_\beta}{r^2} - \frac{1}{2}\delta_{\alpha\beta}\right) + O_2(r^{-2})$$

$$\pi_{\alpha\beta} = \frac{\zeta}{r^2}\frac{x_\alpha x_\beta}{r^2} + \frac{1}{r^3}\left(-B_\alpha x_\beta - B_\beta x_\alpha + (B^\gamma x_\gamma)\delta_{\alpha\beta}\right) + O_1(r^{-3})$$

A, B_α: Constants

$\xi, \zeta \in C^2(\mathbb{S}^2)$

Matching of BMS-supertranslation

- Obtain solutions for background and gauge fields $(\mathbf{e}_a, \hat{\Gamma}_a{}^b{}_c, \hat{L}_{ab}, f_a)$ and the gravitational field $d^a{}_{bcd}$.
- The rescaled Weyl tensor decomposes to $(\phi_0, \phi_1, \phi_2, \phi_3, \phi_4)$
- Transforming to NP-gauge

$$\lim_{\rho\to 0} \sigma^{ab} N_{ab} = 0 \qquad \lim_{\rho\to 0} \Psi_2 = -\phi_2^0, \qquad \text{where } \phi_2 = \phi_2^0 + \mathcal{O}(\rho)$$

Conclusion: The BMS-supertranslation charges at $I^{\pm}$ depends only on T and ϕ_2^0

Matching of BMS-supertranslation

Theorem: For Huang's initial data, Q are not well-defined at the critical sets $I^{\pm}$ unless the initial data satisfy:

$$\xi_{2l,m+l} = 0, \qquad \text{for odd } l \geq 2, \ -l \geq m \geq l$$

If the initial data satisfy these conditions, $Q\,|_{I^{\pm}}$ are fully determined by the initial data and their conservation follows directly from the regularity conditions, specifically we get:

$$Q_{l,m}\,|_{I^{\pm}} \begin{cases} = -21\xi_{0,0} + 6A, & \text{for } l = 0,\ m = 0, \\ = \mp 42\xi_{2,m+1}, & \text{for } l = 1, -1 \leq m \leq 1, \\ = 0, & \text{for odd } l \geq 2, -l \leq m \leq l, \\ \propto -2\xi_{2l,m+l}, & \text{for even } l \geq 2, -l \leq m \leq l, \end{cases}$$

Matching is not generically satisfied unless initial data are fine-tuned!

With fine tuning: $Q_{l,m}\,|_{I^+} = (-1)^l Q_{l,m}\,|_{I^-}$

Other matching results

Similar results for spin-1 and spin-2 fields on Minkowski background

Analysis on n-dimensional spacetime?

[Gasperín, Magdy, Mena (2024)]

- Take $(\mathbb{R}^{1,n-1}, \tilde{\eta})$ and $\tilde{\phi}$ satisfying $\tilde{\Box}\,\tilde{\phi} = 0$
- Introduce coordinate + conformal transformation $(\mathbb{R}^{1,n-1}, \tilde{\eta}) \to (\mathscr{M}, \eta)$:

$$\eta = \Theta^2 \tilde{\eta}, \quad \text{with } \Theta = \rho(1-\tau^2)$$

n-dimensional conformal metric

$$\eta = d\tau \otimes d\tau - \frac{(1-\tau^2)}{\rho^2} d\rho \otimes d\rho + \frac{\tau}{\rho}(d\tau \otimes d\rho + d\rho \otimes d\tau) - \sigma$$

σ : metric on $\mathbb{S}^{n-2}$

Conservation of spin-0 asymptotic charges

- On $(\mathscr{M}, \eta)$: $\qquad \square\phi - \frac{n-2}{4(n-1)}R\phi = 0,\quad$ with $\phi = \Theta^{1-n/2}\tilde{\phi}\quad$ and $R = (n-4)(n-1)$
- Take

$$\phi(\tau, \rho, \theta^A) = \sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m}\frac{1}{p!}a_{p;l,m}(\tau)Y_{l,m}\rho^p,$$

$Y_{l,m}$: spherical harmonics on $\mathbb{S}^{n-2}$

- Substitution

$$(1-\tau^2)\ddot{a}_{p;l,m} + 2\tau(p-1)\dot{a}_{p;l,m} + \frac{1}{4}(2p+n+2l-4)(2l+n-2(1+p))a_{p;l,m} = 0.$$

Conservation of spin-0 asymptotic charges

- Take initial data

$$\phi|_{\mathcal{S}_*}(\rho, \theta^A) = \sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m}\frac{1}{p!}a_{p;l,m}(0)Y_{l,m}\rho^p,$$

$$\dot{\phi}|_{\mathcal{S}_*}(\rho, \theta^A) = \sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m}\frac{1}{p!}\dot{a}_{p;l,m}(0)Y_{l,m}\rho^p,$$

Lemma: Given initial data $\phi|_{\mathcal{S}_*}$ and $\dot{\phi}|_{\mathcal{S}_*}$, the solution to the associated Legendre equation is given by

$$a_{p;l,m}(\tau) = \Big(C_{p;l,m}P^p_\nu(\tau) + D_{p;l,m}Q^p_\nu(\tau)\Big)(1-\tau^2)^{\frac{p}{2}},$$

where $\nu = (1/2)(n-2l-4)$, $C_{p;l,m}$ and $D_{p;l,m}$ are constants related to $a_{p;l,m}(0)$ and $\dot{a}_{p;l,m}(0)$.

Observation: the order ν is integer iff n is even.

Observation: the regularity $a_{p;l,m}(\tau)$ depends on whether ν is integer or not.

P^p_ν : associated Legendre polynomial

Q^p_ν : associated Legendre function of the second kind

Conservation of spin-0 asymptotic charges

- Behaviour of $a_{0;l,m}(\tau)$ at $\tau = \pm 1$ (null infinity)

Spacetime dimension	Legendre Polynomial	Legendre Function	Limit to null infinity of Solution
Even n	Polynomial	Polyhomogeneous	Singular behaviour at past and future null infinity removable by fine-tuning the initial data.
Odd n	Polyhomogeneous	Polyhomogeneous	Singularity at past/ future null infinity. Fine tuning implies trivial solution.

Definition of spin-0 asymptotic charges

[Nguyen & West(2022)], [Henneaux & Troessaert (2019)], [Campiglia & Freidel &Hopfmueller & Soni (2019)]

$$\mathcal{Q}_n(f, \mathscr{C}) = \oint_{\mathscr{C}} f(x^i)\tilde{\phi}^{(n/2-1)}\varepsilon_{n-2},$$

Conservation of spin-0 asymptotic charges

Theorem: Taking initial data $\phi|_{\mathcal{S}_*}$ and $\dot{\phi}|_{\mathcal{S}_*}$, the asymptotic charges $\mathcal{Q}_n(f, \mathscr{C})$ associated with $\tilde{\phi}$ are generically not well-defined at $I^{\pm}$ unless the initial data satisfy extra regularity conditions. Specifically,

1- For even $n \leq 4$, $l \leq 0$, if initial data are chosen so that $D_{p;l,m} = 0$, then $\mathcal{Q}_n(f, \mathscr{C})|_{I^{\pm}}$ are fully written in terms of the initial data

$$\mathcal{Q}_n(Y_{l,m}, \mathscr{I}^{\pm}) = \begin{cases} C_{0;l,m} & \text{for } \tau = 1, \\ (-1)^{\ell} C_{0;l,m} & \text{for } \tau = -1. \end{cases}$$

2- For odd $n > 4$, there exists no non-trivial charges with well-defined limits at $I^{\pm}$.

Matching is not generically satisfied.

For even n, fine tuning implies $\mathcal{Q}_{n;l}|_{I^+} = (-1)^l \mathcal{Q}_{n;l}|_{I^-}$

Conclusions

- Matching of BMS-supertranslation is not generic.
- Future direction: Understand the logarithmic matching of [Fuentealba,Henneaux (2024)] in the context of Friedrich's formulation.

Thank you for listening!