Controlled regularity at future null infinity from past asymptotic initial data: the wave equation

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Virtual Infinity Seminar, Dec 2025

Outline

Motivation

My supervisor told me to?

Follow on from arXiv:2304.08270 [TV23]!

Motivation

It's the 1960s. Lax and Phillips [LP64] have produced a scattering theory for particle interactions using some very formal methods. Within a couple of years, Penrose introduces the compactification of non-compact pseudo-Riemannian manifolds into a compact manifold with boundary describing infinity [Pen63; Pen65].

Fast-forward 20 years. Friedlander has then combined these two completely independent notions together to form what is known as conformal scattering theory [Fri62; Fri64; Fri80].

With the notion of the conformal boundary now available, it raises a question: *how is the past and future asymptotic data of massless fields related to each other?*

Motivation

Take an asymptotically flat pseudo-Riemannian manifold. Then the conformal boundary $\mathscr{I}$ is a null hypersurface. So, a conformal scattering problem here takes the form of a characteristic initial value problem with data prescribed on $\mathscr{I}$ [Pen80; Fri80].

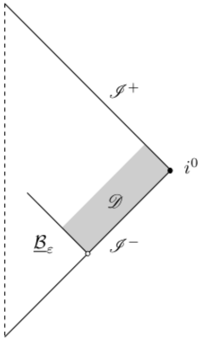


Figure: The domain of interest for the wave equation.

Motivation

- Refining our question from earlier: *how does the gravitational radiation of physical objects and the regularity of the past conformal boundary $\mathscr{I}^-$ affect the structure of solutions at the future $\mathscr{I}^+$?*
- One answer: There is polyhomogeneous behaviour towards $\mathscr{I}^+$.

  If $\Omega \geq 0$ denotes the boundary defining function for the manifold M with $\Omega^{-1}(0) = \partial M$, then solutions contain terms in their expansions towards the boundary which look like $\Omega^\alpha \log^\beta \Omega$ [Fri98b; CK93; LR10; HV17; Lin17; KK25].

  Einstein's field equations are very nonlinear and are partially responsible for this.

Geometric set-up

Here, we encounter *the problem of spatial infinity* [Fri98b; Val04a; Val04b; GV17; DF17; BDFW12].

Studying the wave equation as a characteristic initial value problem with data on past null infinity $\mathscr{I}^-$ implies that we need a better representation of spatial infinity.

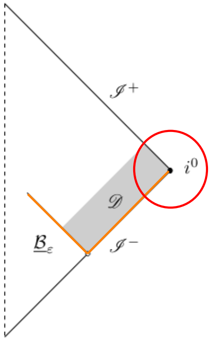


Figure: The domain of interest for the wave equation.

Geometric set-up

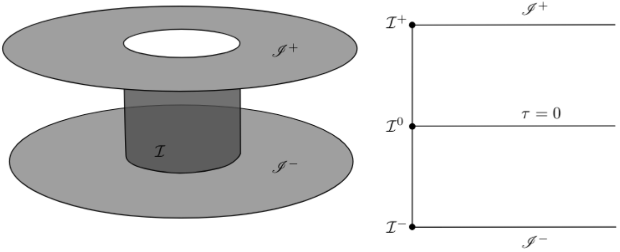


Figure: Friedrich's cylinder at spatial infinity [Fri98a].

The coordinate transformation

$$\rho = \frac{r}{r^2 - t^2}, \quad \tau = \frac{t}{r}, \quad \left(\Omega = \rho(1+\tau)(1-\tau) \equiv \frac{1}{r}\right)$$

and the geometric blow-up together give what we call the *F-gauge*.

Main theorem

Theorem (rough version)

Solutions to the wave equation near spatial infinity in the Minkowski spacetime with sufficiently regular asymptotic characteristic initial data at past null infinity $\mathscr{I}^-$ and a short incoming null hypersurface $\mathcal{B}_\varepsilon$ possess suitably regular asymptotic expansions in a neighbourhood of spatial infinity i^0, and in particular exhibit peeling at future null infinity $\mathscr{I}^+$.

Formulation of the problem

- Re-cast the wave equation on Minkowski space $(\mathbb{R}_t \times \mathbb{R}^3_x, \tilde{\boldsymbol{\eta}})$,

$$\Box_{\tilde{\boldsymbol{\eta}}}\tilde{\phi} \equiv \left(\partial_t^2 - \sum_{i=1}^{3}\partial_{x^i}^2\right)\tilde{\phi} = 0,$$

as a symmetric hyperbolic system [Kat75] (cf. [Ren90; Luk12]) in the F-gauge.

$$\Box_{\boldsymbol{\eta}}\phi = (1-\tau^2)\partial_\tau^2\phi + 2\tau\rho\partial_\tau\partial_\rho\phi - \rho^2\partial_\rho^2\phi - 2\tau\partial_\tau\phi - \Delta\!\!\!/_{\mathbb{S}^2}\phi = 0.$$

- Defining the variables,

$$\psi = \sqrt{2}\partial_\tau\phi, \qquad \psi_0 = \frac{1}{\sqrt{2}}\boldsymbol{X}_-\phi,$$
$$\psi_1 = -\frac{1}{\sqrt{2}}(\tau\partial_\tau\phi + \rho\partial_\rho\phi), \qquad \psi_2 = -\frac{1}{\sqrt{2}}\boldsymbol{X}_+\phi,$$

so that we have the system of equations to find some related conserved quantities...

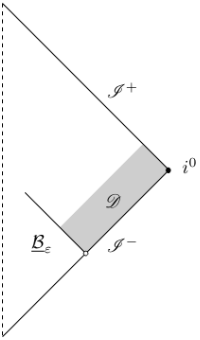
Formulation of the problem

- The equations one gets from the symmetric hyperbolic system in terms of the auxiliary variables are as follows

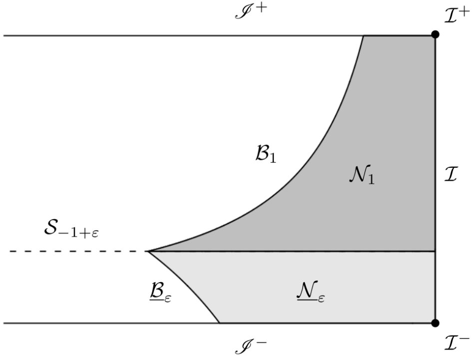
$$\begin{aligned}
A_0 &\equiv (1+\tau)\partial_\tau \psi_2 - \rho\partial_\rho \psi_2 - \frac{1}{2}\boldsymbol{X}_-\psi - \boldsymbol{X}_-\psi_1 = 0, \\
B_0 &\equiv (1-\tau)\partial_\tau \psi_1 + \rho\partial_\rho \psi_1 + \frac{1}{2}\left((1-\tau)\partial_\tau\psi + \rho\partial_\rho\psi\right) \\
&\quad - \boldsymbol{X}_+\psi_2 - \frac{1}{2}\psi - \psi_1 = 0, \\
A_1 &\equiv (1+\tau)\partial_\tau \psi_1 - \rho\partial_\rho \psi_1 - \frac{1}{2}\left((1+\tau)\partial_\tau\psi - \rho\partial_\rho\psi\right) \\
&\quad - \boldsymbol{X}_-\psi_0 - \frac{1}{2}\psi + \psi_1 = 0, \\
B_1 &\equiv (1-\tau)\partial_\tau \psi_0 + \rho\partial_\rho \psi_0 + \frac{1}{2}\boldsymbol{X}_+\psi - \boldsymbol{X}_+\psi_1 = 0.
\end{aligned}$$

- Applying the operator $D \equiv \partial_\rho^p \partial_\tau^q \boldsymbol{Z}^\alpha$ to each equation in and multiplying by $\overline{D\psi_k}$ and $\overline{D\psi}$ in an appropriate combination yields the following higher-order currents,

$$\begin{aligned}
0 &= 2\,\mathrm{Re}\left(\overline{D\psi_2}DA_0 + \overline{D\psi_1}DB_0\right) + \mathrm{Re}\left(\overline{D\psi}DB_0\right), \\
0 &= 2\,\mathrm{Re}\left(\overline{D\psi_1}DA_1 + \overline{D\psi_0}DB_1\right) - \mathrm{Re}\left(\overline{D\psi}DA_1\right).
\end{aligned}$$



$\mathscr{I}^+$
i^0
$\mathscr{D}$
$\underline{\mathcal{B}}_\varepsilon$
$\mathscr{I}^-$



$\mathscr{I}^+$
$\mathcal{I}^+$
$\mathcal{B}_1$
$\mathcal{N}_1$
$\mathcal{I}$
$\mathcal{S}_{-1+\varepsilon}$
$\underline{\mathcal{B}}_\varepsilon$
$\underline{\mathcal{N}}_\varepsilon$
$\mathscr{I}^-$
$\mathcal{I}^-$

Main results

Theorem 1

Let $\rho_\star > 0, 0 < \varepsilon \ll 1$ be real numbers and $m \in \mathbb{N}$ be an integer. Given data on the past conformal boundary $\mathscr{I}^-$ and on a short incoming null hypersurface that is sufficiently regular, the wave equation admits a unique solution with the expansion

$$\phi = \sum_{p'=0}^{m+4} \frac{1}{p'!} \phi^{(p')} \left(\tau, t^{\boldsymbol{A}}{}_{\boldsymbol{B}}\right) \rho^{p'} + C^{m,\alpha}$$

at the future conformal boundary $\mathscr{I}^+$ where $0 < \alpha \leq \frac{1}{2}$.

The τ-dependence of the coefficients of the expansion can be computed explicitly in terms of solutions to Jacobi ODEs [Sze78].

Main results

Theorem 1 (technical version)

Let $\rho_* > 0$, $0 < \varepsilon \ll 1$ be real numbers and $m \in \mathbb{N}$ be an integer. Suppose the asymptotic characteristic data for $\Box\phi = 0$ for the components $f \in \{\psi, \psi_0, \psi_1, \psi_2\}$ on $\mathscr{I}^-_{\rho_\star} \cup \underline{\mathcal{B}}_\varepsilon$ has the regularity

$$(f, \partial_{\nparallel} f, \ldots, \partial_{\nparallel}^{4m+23} f) \in H^{4m+23} \times H^{4m+22} \times \ldots \times L^2, \qquad (6.1)$$

where $\partial_{\nparallel}$ denotes a transverse derivative to $\mathscr{I}^-$ or $\underline{\mathcal{B}}_\varepsilon$, i.e. $\partial_{\nparallel} = \partial_\tau$ on $\mathscr{I}^-$ and $\partial_{\nparallel} = \partial_\rho$ on $\underline{\mathcal{B}}_\varepsilon$. Additionally, suppose that

$$\phi|_{\mathscr{I}^-} \in H^{4m+24}(\mathscr{I}^-_{\rho_\star}) \quad \text{and} \quad \phi|_{\underline{\mathcal{B}}_\varepsilon} \in H^{4m+24}(\underline{\mathcal{B}}_\varepsilon). \qquad (6.2)$$

Then, in the domain $\mathscr{D} \equiv \underline{\mathcal{N}}_\varepsilon \cup \mathcal{N}_1$, this data gives rise to a unique solution to the wave equation which near $\mathscr{I}^+$ admits the Taylor-like expansion from before.

Main results

Theorem 2

Under the assumptions of Theorem 1, the expansion of the solution

$$\sum_{p'=0}^{m+4} \frac{1}{p'!} \phi^{(p')} \left(\tau, t^{\boldsymbol{A}}{}_{\boldsymbol{B}}\right) \rho^{p'}$$

does not contain logarithmic divergences at $\tau = \pm 1$. In fact, these terms are analytic in τ at $\tau = \pm 1$.

Question: Were you paying attention?

- This was not the generic behaviour predicted (solutions are not polyhomogeneous)! We can still obtain a "nice" class of solutions, i.e. solutions which peel in physical coordinates on the non-compact manifold, under these assumptions.
- These results subsume the physical assumption, the no-incoming radiation condition [Som12; Som92; Mad70], from $\mathscr{I}^-$.

Main results

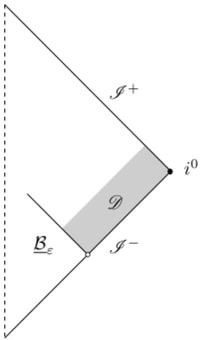
Or in coordinates on the physical spacetime,

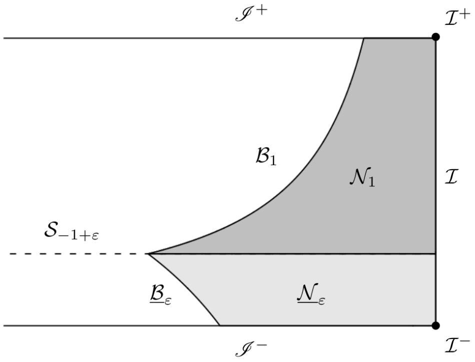
$$\tau = 1 + \frac{u}{r}, \qquad \rho = -\frac{1}{u\left(2 + \frac{u}{r}\right)},$$

$$\tilde{\phi} = \frac{1}{r}\left(\tilde{\varphi}^{(0)}(1, t^{\boldsymbol{A}}{}_{\boldsymbol{B}}) + \sum_{p=0}^{m+4} \frac{1}{p!}\left(\frac{-1}{2u}\right)^{p} \varphi^{(p)}(1, t^{\boldsymbol{A}}{}_{\boldsymbol{B}}) + \mathcal{O}\left(\frac{1}{u^{m+5}}\right)\right).$$

Sketch of the proof

Recall the setting we wish to understand.





On conserved quantities...

Expand the first current

$$0 = 2\,\mathrm{Re}\left(\overline{D\psi_2}DA_0 + \overline{D\psi_1}DB_0\right) + \mathrm{Re}\left(\overline{D\psi}DB_0\right)$$

to see some recurring structure to make some arguments easier.

$$\begin{aligned}
0 = \begin{pmatrix}\partial_\tau \\ \partial_\rho\end{pmatrix} \cdot \begin{pmatrix} (1+\tau)|D\psi_2|^2 + (1-\tau)|D\psi_1|^2 + \tfrac{1}{4}(1-\tau)|D\psi|^2 + (1-\tau)\,\mathrm{Re}\left(\overline{D\psi}D\psi_1\right) \\ -\rho|D\psi_2|^2 + \rho|D\psi_1|^2 + \tfrac{1}{4}\rho|D\psi|^2 + \rho\,\mathrm{Re}\left(\overline{D\psi}D\psi_1\right) \end{pmatrix} \\
- \boldsymbol{Z}^\alpha \boldsymbol{X}_+(\partial_\rho^p\partial_\tau^q\psi_2)\boldsymbol{Z}^\alpha(\partial_\rho^p\partial_\tau^q\overline{\psi}_1) - \boldsymbol{Z}^\alpha(\partial_\rho^p\partial_\tau^q\psi_2)\boldsymbol{Z}^\alpha\boldsymbol{X}_+(\partial_\rho^p\partial_\tau^q\overline{\psi}_1) \\
- \boldsymbol{Z}^\alpha \boldsymbol{X}_-(\partial_\rho^p\partial_\tau^q\psi_1)\boldsymbol{Z}^\alpha(\partial_\rho^p\partial_\tau^q\overline{\psi}_2) - \boldsymbol{Z}^\alpha(\partial_\rho^p\partial_\tau^q\psi_1)\boldsymbol{Z}^\alpha\boldsymbol{X}_-(\partial_\rho^p\partial_\tau^q\overline{\psi}_2) \\
- \frac{1}{2}\left(\boldsymbol{Z}^\alpha \boldsymbol{X}_+(\partial_\rho^p\partial_\tau^q\psi_2)\boldsymbol{Z}^\alpha(\partial_\rho^p\partial_\tau^q\overline{\psi}) + \boldsymbol{Z}^\alpha(\partial_\rho^p\partial_\tau^q\psi_2)\boldsymbol{Z}^\alpha\boldsymbol{X}_+(\partial_\rho^p\partial_\tau^q\overline{\psi})\right) \\
- 2(p-q)|D\psi_2|^2 + 2(p-q-1)|D\psi_1|^2 + \frac{1}{2}(p-q-1)|D\psi|^2 + 2(p-q-1)\,\mathrm{Re}\left(\overline{D\psi}D\psi_1\right).
\end{aligned}$$

On expansions...

- Once you run the energy estimates using the above scheme, one can obtain that, for instance, that p ρ-derivatives of the auxiliary variables belong to a Sobolev space which one can embed into some Hölder spaces losing 3 derivatives in our case.
- For $k \in \{0, 1, 2\}$ and $p > m + 1$,

$$\partial_\rho^p \psi,\ \partial_\rho^p \psi_k \in H^m(\mathcal{N}_1) \hookrightarrow C^{r,\alpha}(\mathcal{N}_1),$$

for r a positive integer and $\alpha \in (0, 1)$ satisfying $r + \alpha = m - \frac{5}{2}$ and $m \geq 3$. Equivalently, $m \geq r + \alpha + \frac{5}{2}$. Restricting to $\alpha \leq \frac{1}{2}$, we have

$$\partial_\rho^p \psi,\ \partial_\rho^p \psi_k \in H^{r+3}(\mathcal{N}_1) \hookrightarrow C^{r,\alpha}(\mathcal{N}_1)$$

whenever $p > m + 1 \geq r + 4$.

Future directions

- A similar result of this kind would be excellent to understand on a black hole background, to see what effects the curvature has on the estimates used to prove such a result (cf. [Mas22; Mas24; Keh24]). And, to see how much the conclusion of the theorems change, i.e. can sufficiently smooth data give rise to polyhomogeneous behaviour?
- One may ask if the set-up/tools was restricted from the beginning [MV21]. To this end, we may approach this problem in a new manner by combining the b-calculus that is used in the school of Melrose (notable texts [Mel95; HV17]) together with Friedrich's cylinder to gain deeper insights into asymptotic behaviour in this corner of the compact manifold.
  This is ongoing work :)
- Release a new paper from Overleaf prison.

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Thank you for listening! Are there any questions?

The wave equation as a symmetric hyperbolic system

$$\mathcal{D}\phi = \psi,$$

$$\mathcal{D}\psi + 2\mathcal{D}^{\boldsymbol{AB}}\psi_{\boldsymbol{AB}} = 0,$$

$$\frac{4}{(\boldsymbol{A}+\boldsymbol{B})!(2-\boldsymbol{A}-\boldsymbol{B})!}\left(\mathcal{D}\psi_{\boldsymbol{AB}} - \mathcal{D}_{\boldsymbol{AB}}\psi + 2\mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\psi_{\boldsymbol{B})\boldsymbol{Q}}\right) = 0,$$