Construction of hyperboloidal initial data without logarithmic singularities

Rácz István
racz.istvan@wigner.hu
HUN-REN Wigner RCP

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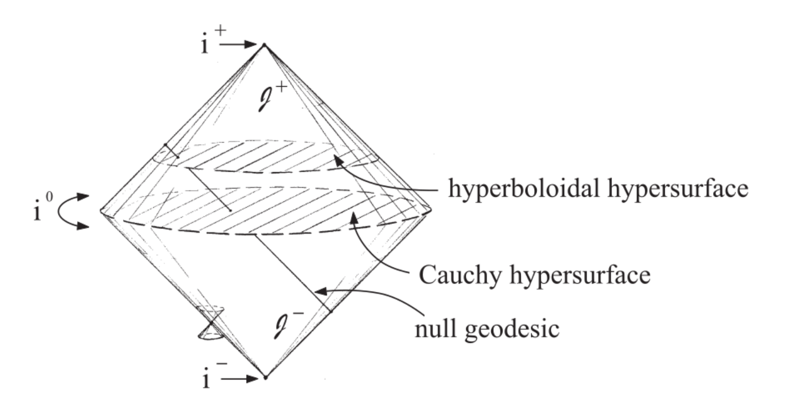
Penrose 1963: asymptotically simple spacetimes & conformal compactification

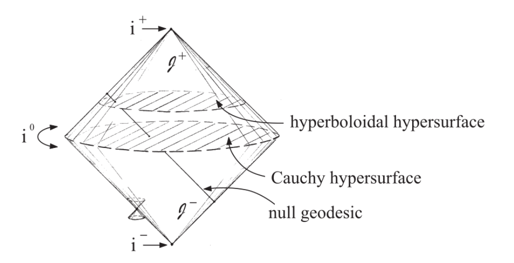


the asymptotic behavior of the gravitational field of an isolated system resembles that of Minkowski space [Image credit to H. Friedrich]

- There exists a smooth manifold $\widetilde{M} = M \cup \partial M$ with boundary,
- and a smooth Lorentz metric $\widetilde{g} = \Omega^2 g$ on $\widetilde{M}$
- such that $\Omega = 0$, $d\Omega \neq 0$ on ∂M
    - The **degree of smoothness** of the involved structures **is critical**.

Asymptotically simple spacetimes & asymptotically hyperboloidal data

- **Friedrich:** Solutions to the **hyperboloidal initial value formulation of his conformal field equations** are asymptotically simple in the future.
  - Hyperboloidal initial data – sufficiently and close to Minkowskian hyperboloidal data – develop into solutions which admit smooth conformal extensions containing a regular point i^+ that represents future time-like infinity.



the asymptotic behavior of the gravitational field of an isolated system resembles that of Minkowski space [Image credit to H. Friedrich]

- An **asymptotically hyperboloidal vacuum** data set (Σ, h_{ij}, K_{ij}) is such that
  - Σ is the interior of a **compact manifold with boundary** $\Sigma \cup \partial\Sigma$
  - the **trace** $K = K_{ij}h^{ij}$ of K_{ij} is **bounded away from zero** near $\partial\Sigma$
  - if ω is a defining function for $\partial\Sigma$ then $\omega^2 h_{ij}$ and $\omega \mathring{K}_{ij}$ **extend regularly** to $\partial\Sigma$

Constraint equations

Einstein equations: $E_{ab} = G_{ab} - \mathscr{G}_{ab} = 0$ $\quad n^a n^b E_{ab} = 0$ & $\pi_i{}^a n^b E_{ab} = 0$

- in the vacuum case ($\mathscr{G}_{ab} = 0$) for (h_{ij}, K_{ij}) on Σ

$$\boxed{{}^{(3)}R + \left(K_{ij}h^{ij}\right)^2 - K_{ij}K^{ij} = 0 \quad \& \quad D^i\left[K_{ij} - h_{ij}\left(K_{ef}h^{ef}\right)\right] = 0} \quad (D_k h_{ij} = 0)$$

- It is an underdetermined system, 4 equations for the 12 variables: $\boxed{(h_{ij}, K_{ij})}$

Conformal method (Lichnerowicz 1944, York 1972)

$$h_{ij} = \phi^4\, \widetilde{h}_{ij}, \qquad \mathring{K}_{ij} = K_{ij} - \tfrac{1}{3}\, h_{ij} K = \phi^{-2}\, \widetilde{K}_{ij}, \qquad \widetilde{K}_{ij} = CKO[X]_{ij} + \widetilde{K}_{ij}^{[TT]}$$

$$(h_{ij}, K_{ij}) \quad \longleftrightarrow \quad (\phi, \widetilde{h}_{ij}; K, X_i, \widetilde{K}_{ij}^{[TT]})$$

the constraints form a quasilinear elliptic system for (ϕ, X_i) [!boundary value]

Evolutionary method (I.R. 2015)

Σ is foliated by topological two-spheres & applying an ADM type decomposition

$$h_{ij} \longleftrightarrow \widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij} \quad \& \quad K_{ij} \longleftrightarrow \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}_{ij} \, [= \mathring{\mathbf{K}}_{ij} + \tfrac{1}{2}\, \mathbf{K}\, \widehat{\gamma}_{ij}]$$

using a complex null dyad q^i: $q_{ij} = q_{(i}\overline{q}_{j)}$ $\qquad (h_{ij}; K_{ij}) \longleftrightarrow (\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \mathring{\mathbf{K}}_{qq})$

the constraints form a parabolic-hyperbolic system for $(\widehat{\mathbf{N}}, \mathbf{k}, \mathbf{K})$ [!initial data]

Andersson & Chruściel '93, '94, '96: log-terms become part of the discussions

- They proved that **even if the free data is:** $\omega^2 h_{ij}$ & $\omega \mathring{K}_{ij}$ are regular on $\Sigma \cup \partial\Sigma$,
  (1) **in general**, the constrained fields have poly-logarithmic expansions in $\omega \sim \rho^{-1}$, where ρ denotes the "distance" from the isolated system

$$\boxed{\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1\,\omega + \mathcal{C}_2\,\omega^2 + \ldots \qquad \mathcal{C} = \mathcal{C}_0 + \textstyle\sum_{i=1}^{\infty} \omega^i \left[\mathcal{C}_i + \sum_{j=1}^{\mathcal{N}_j} \mathcal{C}_{i,j}^{[log]} \log^j \omega \right]}$$

$$\Uparrow \qquad (n \geq 1) \quad \boxed{\omega \log^n \omega \to 0 \quad \textbf{but} \quad \partial_\omega(\omega \log^n \omega)} \text{ blows up}$$

  (2) **non-generic cases:** the initial data can be smooth (free of log-terms) on $\widetilde{\Sigma}$ if the relations

$$\boxed{\widetilde{K}^{[log]}_{ra}|_{\partial\Sigma} = 0} \text{ in } \widetilde{K}_{ab} = \widetilde{K}^{[C^\infty]}_{ab} + \Omega^2 \log\Omega \cdot \widetilde{K}^{[log]}_{ab} \quad \& \quad \boxed{(\widetilde{\mathbf{K}}^{\circ}_{ab} - \widetilde{\widetilde{K}}^{\circ}_{ab})|_{\partial\Sigma} = 0}$$

  (derived from the free data) **hold on** $\partial\Sigma$.

Facing the problems:

- These conclusions were disappointing because they implied that the initial data constructed by the conformal method is **generally not regular enough** for use in Friedrich's existence theorems. $[\widetilde{\nabla}_e d_{abc}{}^e = 0 \;\; \& \;\; d_{abc}{}^e = \Omega^{-1} C_{abc}{}^e]$
- **If the initial data involves log-terms**, then **the development will also contain**.
- How can then the metric be decomposed into a sufficiently smooth non-physical metric and a conformal factor? the concept of asymptotically simple spacetimes?

Important new results by Beyer and Ritchie [CQG,**39**,145012 (2022)]

- They assumed that there exist smooth global solutions to the **parabolic-hyperbolic form of the constraints** on a "hyperboloidal initial data surface" Σ.
- They also assumed that these solutions **extend regularly up to order** 4 **and** 3, respectively, to $\partial\Sigma$
  - $\widehat{\mathbf{N}}, \mathbf{K} \in C^4\big([0,\omega_0), C^\infty(\mathscr{S}^2)\big)$
  - $\mathbf{k} \in C^3\big([0,\omega_0), C^\infty(\mathscr{S}^2)\big)$
- Then, using an **impressive Fuchsian-equation based argument** they showed that **the constrained variables extend smoothly** to $\partial\Sigma$, whence such solutions are free of all log-terms.
- **parabolic-hyperbolic form of the constraints** ✓
  - BUT **even stronger** assumptions than used by Andersson and Chruściel.
  - They restricted not only the free data (that went largely uncommented)
  - **but** also **the constrained fields** $\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k}$ **!!!**

The strategy we used:

Asymptotically hyperboloidal data: $(h_{ij}; K_{ij}) \longleftrightarrow (\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \mathring{\mathbf{K}}_{qq})$

- Note first that a data set $(\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \mathring{\mathbf{K}}_{qq})$ can only be **asymptotically hyperboloidal** $[\omega^2 h_{ij}\ \&\ \omega \mathring{K}_{ij}]$ if the following falloff conditions hold: $\omega \sim \rho^{-1}$

$$\widehat{\mathbf{N}} = \widehat{\mathbf{N}}_1 \omega + \mathscr{O}(\omega^2) \qquad \mathbf{K} - 2\,\boldsymbol{\kappa} = \mathscr{O}(\omega) \qquad \mathbf{k} = \mathscr{O}(1)$$

$$\mathbf{a} = \omega^{-2} + \mathscr{O}(\omega^{-1}) \qquad \mathbf{b} = \mathscr{O}(\omega^{-1}) \qquad \mathbf{N} = \mathscr{O}(\omega) \qquad \mathring{\mathbf{K}}_{qq} = \mathscr{O}(\omega^{-1})$$

The free data is assumed to be: $\omega^2 h_{ij}\ \&\ \omega \mathring{K}_{ij}$ regular on $\Sigma \cup \partial\Sigma$ [no log-terms!]

$$\begin{aligned}
\mathbf{N} &= \mathbf{N}_1\,\omega + \mathbf{N}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathbf{a} &= \omega^{-2} + \mathbf{a}_{(-1)}\,\omega^{-1} + \mathbf{a}_0 + \mathbf{a}_1\,\omega + \mathbf{a}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathbf{b} &= \mathbf{b}_{(-1)}\,\omega^{-1} + \mathbf{b}_0 + \mathbf{b}_1\,\omega + \mathbf{b}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\boldsymbol{\kappa} &= \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1\,\omega + \boldsymbol{\kappa}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathring{\mathbf{K}}_{qq} &= \mathring{\mathbf{K}}_{qq\,(-1)}\,\omega^{-1} + \mathring{\mathbf{K}}_{qq\,0} + \mathring{\mathbf{K}}_{qq\,1}\,\omega + \mathring{\mathbf{K}}_{qq\,2}\,\omega^2 + \mathscr{O}(\omega^3)
\end{aligned}$$

Use the most generic poly-logarithmic form of the constrained fields $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$:

$$\widehat{\mathbf{N}} = \sum_{i=1}^{\infty} \omega^i \left[\, \widehat{\mathbf{N}}_i + \sum_{j=1}^{\mathcal{N}_j} \widehat{\mathbf{N}}_{i,j}^{[log]} \log^j \omega \,\right], \qquad \mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{\infty} \omega^i \left[\, \mathbf{K}_i + \sum_{j=1}^{\mathcal{N}_j} \mathbf{K}_{i,j}^{[log]} \log^j \omega \,\right]$$

$$\mathbf{k} = \mathbf{k}_0 + \sum_{i=1}^{\infty} \omega^i \left[\, \mathbf{k}_i + \sum_{j=1}^{\mathcal{N}_j} \mathbf{k}_{i,j}^{[log]} \log^j \omega \,\right], \text{where } \widehat{\mathbf{N}}_1 = \boldsymbol{\kappa}_0^{-1}, \mathbf{K}_0 = 2\boldsymbol{\kappa}_0, \mathbf{k}_0 = \boldsymbol{\kappa}_0^{-1} \eth \boldsymbol{\kappa}_0$$

Our first main result

Theorem I:

- Choose **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{qq})$ on Σ such that it **satisfies the asymptotically hyperboloidal falloff conditions**, i.e., $\omega^2 h_{ij}$ & $\omega \mathring{K}_{ij}$ are **regular** on $\Sigma \cup \partial\Sigma$.
- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth solutions** of the parabolic-hyperbolic form of the constraints on Σ.
- $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are also assumed to possesses the most generic poly-logarithmic expansion near $\partial\Sigma$ as indicated above.
- Then the corresponding asymptotically hyperboloidal initial data set **admits well-defined Bondi energy and angular momentum** **if and only if** all coefficients of **the logarithmic terms vanish up to order four and three** for $\widehat{\mathbf{N}}, \mathbf{K}$ and $\mathbf{k}$, respectively, and, **in addition,**

$$\mathring{\mathbf{K}}_{qq\,(-1)} = 0\,, \quad \mathbf{b}_{(-1)} = 0\,, \quad \boldsymbol{\kappa}_1 = 0\,.$$

$$\mathbf{b} = \cancel{\mathbf{b}_{(-1)}}\,\omega^{-1} + \mathbf{b}_0 + \mathbf{b}_1\,\omega + \mathbf{b}_2\,\omega^2 + \mathscr{O}(\omega^3)$$
$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_0 + \cancel{\boldsymbol{\kappa}_1}\,\omega + \boldsymbol{\kappa}_2\,\omega^2 + \mathscr{O}(\omega^3)$$
$$\mathring{\mathbf{K}}_{qq} = \cancel{\mathring{\mathbf{K}}_{qq\,(-1)}}\,\omega^{-1} + \mathring{\mathbf{K}}_{qq\,0} + \mathring{\mathbf{K}}_{qq\,1}\,\omega + \mathring{\mathbf{K}}_{qq\,2}\,\omega^2 + \mathscr{O}(\omega^3)$$

The finiteness of the Bondi energy:

The Bondi energy can be given as the $\rho \to \infty$ limit of the Hawking energy

$$E_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \left(1 + \frac{1}{16\pi} \int_{\mathscr{S}_\rho} \underbrace{\left(\mathbf{K}^2 - \overset{\star}{\mathbf{K}}{}^2 \widehat{\mathbf{N}}^{-2} \right)}_{= 2\,\theta^{(+)}\theta^{(-)} = -4\left[\,2Re(\psi_2 - \sigma\sigma')\,\right] - 2\widehat{R}} \sqrt{\mathbf{d}}\, \overset{\circ}{\boldsymbol{\epsilon}} \right)$$

$\mathcal{A}, \sqrt{\mathbf{d}} \sim \rho^2$, $\overset{\star}{\mathbf{K}} = \partial_\rho \log[\sqrt{\mathbf{d}}] \sim \rho^{-1}$ requires $\int_{\mathscr{S}_\rho} \left(\mathbf{K}^2 - \overset{\star}{\mathbf{K}}{}^2 \widehat{\mathbf{N}}^{-2} \right) \sqrt{\mathbf{d}}\, \overset{\circ}{\boldsymbol{\epsilon}} = -16\pi + \mathcal{O}(\rho^{-1})$

$\lim_{\rho\to\infty} E_H$ is finite $\Longrightarrow$ for all $j = 1, 2, \dots, \mathcal{N}_j$

$$\widehat{\mathbf{N}}{}^{[log]}_{1,j} = \widehat{\mathbf{N}}{}^{[log]}_{2,j} = \mathbf{K}^{[log]}_{1,j} = 0$$

$$\mathbf{K}^{[log]}_{2,i} = \mathbf{K}_0 \left(2\,\mathbf{K}^{[log]}_{3,i} + \mathbf{K}_0^2\, \widehat{\mathbf{N}}{}^{[log]}_{4,i} \right) \cdot \left[\, \mathbf{a}_{(-1)}\, \mathbf{K}_0 + 4\,\mathbf{K}_1 \,\right]^{-1}$$

$$\widehat{\mathbf{N}}{}^{[log]}_{3,i} = -2 \left(2\,\mathbf{K}^{[log]}_{3,i} + \mathbf{K}_0^2\, \widehat{\mathbf{N}}{}^{[log]}_{4,i} \right) \cdot \left(\mathbf{K}_0 \left[\, \mathbf{a}_{(-1)}\, \mathbf{K}_0 + 4\,\mathbf{K}_1 \,\right] \right)^{-1}$$

The finiteness of the Bondi angular momentum:

- details of the construction in I.R. Phys. Rev. D 112, 064044 (2025)
- $\rho \to \infty$ limits of the integral expressions for any axial vector field ϕ^a

$$J[\phi] = -(8\,\pi)^{-1} \int_{\mathscr{S}_\rho} \left(\phi^a [\overline{q}_a \mathbf{k} + q_a \overline{\mathbf{k}}] \right) \sqrt{\mathbf{d}}\, \overset{\circ}{\boldsymbol{\epsilon}}$$

- since $\sqrt{\mathbf{d}} \sim \rho^2$ the Bondi angular momentum cannot be finite unless for all $j = 1, 2, \dots, \mathcal{N}_j$

$$\mathbf{k}^{[log]}_{1,j} = \mathbf{k}^{[log]}_{2,j} = 0$$

Completion of the proof

- To obtain the desired restrictions, we substitute the **updated form** of the asymptotic expansions into the **parabolic-hyperbolic system** and sort the terms with respect to powers of ρ^{-1} and also of $\log\,\rho$. $\Longrightarrow$

$$\mathbf{k}^{[log]}_{1,j} = \mathbf{k}^{[log]}_{2,j} = \mathbf{k}^{[log]}_{3,j} = 0\,, \qquad \begin{array}{l} \widehat{\mathbf{N}}^{[log]}_{1,j} = \widehat{\mathbf{N}}^{[log]}_{2,j} = \widehat{\mathbf{N}}^{[log]}_{3,j} = \widehat{\mathbf{N}}^{[log]}_{4,j} = 0 \\ \mathbf{K}^{[log]}_{1,j} = \mathbf{K}^{[log]}_{2,j} = \mathbf{K}^{[log]}_{3,j} = \mathbf{K}^{[log]}_{4,j} = 0 \end{array}$$

Inspecting the consequences

- The above findings suggest using the **asymptotic expansions** for constrained variables:
  - $\widehat{\mathbf{N}} \longrightarrow \widehat{\mathbf{N}}_0 + \widehat{\mathbf{N}}_1\,\omega + \widehat{\mathbf{N}}_2\,\omega^2 + \widehat{\mathbf{N}}_3\,\omega^3 + \widehat{\mathbf{N}}_4\,\omega^4 + \omega^4 w_{\widehat{\mathbf{N}}}(\omega)$
  - $\mathbf{K} \longrightarrow \mathbf{K}_0 + \mathbf{K}_1\,\omega + \mathbf{K}_2\,\omega^2 + \mathbf{K}_3\,\omega^3 + \mathbf{K}_4\,\omega^4 + \omega^4 w_{\mathbf{K}}(\omega)$
  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$
- The functions $w_{\widehat{\mathbf{N}}}(\omega), w_{\mathbf{K}}(\omega), w_{\mathbf{k}}(\omega)$, are of class $C^0\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ and **vanish at** $\partial\Sigma$, thus they can represent **all the higher-order log-terms** that may still occur.
- The coefficients $(\widehat{\mathbf{N}}_4, \mathbf{K}_1, \mathbf{k}_2)$ represent the **asymptotic degrees of freedom** since
  - All of the "**bold-faced coefficients in black**" can be **derived from the free data**, $(\mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$, and from the asymptotic degrees of freedom $(\widehat{\mathbf{N}}_4, \mathbf{K}_1, \mathbf{k}_2)$.

The asymptotic degrees of freedom & free data determine:

- $(\widehat{\mathbf{N}}_4, \mathbf{K}_1, \mathbf{k}_2)$, & **free data**, $(\mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{qq})$: The **Bondi energy**

$$\boxed{\begin{aligned} E_B = \frac{1}{32\pi}\int_{\partial\Sigma} \Big[& \frac{{\mathbf{K}_1}^3}{\boldsymbol{\kappa}_0} + 8\,\widehat{\mathbf{N}}_4\,{\boldsymbol{\kappa}_0}^3 - 4\,\boldsymbol{\kappa}_0\,\boldsymbol{\kappa}_3 + 12\,\mathbf{a}_1\,{\boldsymbol{\kappa}_0}^2 - 2\,\mathbf{K}_1\left[{\boldsymbol{\kappa}_0}^{-1} + 3\,\mathbf{a}_0\,\boldsymbol{\kappa}_0 - 4\,\boldsymbol{\kappa}_2\right] \\ & - \Big\{ {\boldsymbol{\kappa}_0}^5\,{\mathbf{N}_1}^{-1/2}\,\bar{\eth}\left({\mathbf{N}_1}^{3/2}{\boldsymbol{\kappa}_0}^{-4}\mathbf{K}_1\right) + 2\,\boldsymbol{\kappa}_0\,{\mathbf{N}_2}^2\,\bar{\eth}\left({\mathbf{N}_2}^{-1}\boldsymbol{\kappa}_0\right) + \tfrac{1}{4}\,{\boldsymbol{\kappa}_0}^5\,\eth\bar{\eth}\left({\boldsymbol{\kappa}_0}^{-6}\,\mathbf{K}_1\right) \\ & - \tfrac{3}{2}\,{\boldsymbol{\kappa}_0}^{-2}\,\mathbf{K}_1\,\eth\bar{\eth}\,\boldsymbol{\kappa}_0 + \tfrac{1}{4}\,\boldsymbol{\kappa}_0\,\eth\bar{\eth}\,\mathbf{K}_1 + \text{“}cc\text{”} \Big\} \Big] \mathring{\boldsymbol{\epsilon}} \end{aligned}}$$

& the components of the **Bondi angular momentum**

$$\boxed{J_B[\mathring{\phi}_{(i)}] = -(16\,\pi)^{-1}\int_{\partial\Sigma}\left[(\mathring{\phi}^a_{(i)}\overline{q}_a)\,\mathbf{k}_2 + (\mathring{\phi}^a_{(i)}q_a)\,\overline{\mathbf{k}}_2\right]\mathring{\boldsymbol{\epsilon}}}$$

  - where $\phi^a_{(i)}$, $(i = 1, 2, 3)$, denote the **axial Killing vector fields**

$$\boxed{{\mathring{\phi}_{(1)}}^a = -\sin\varphi\,(\partial_\vartheta)^a - \cot\vartheta\,\cos\varphi\,(\partial_\varphi)^a \;\;\&\;\; {\mathring{\phi}_{(2)}}^a = \cos\varphi\,(\partial_\vartheta)^a - \cot\vartheta\,\sin\varphi\,(\partial_\varphi)^a \;\;\&\;\; {\mathring{\phi}_{(3)}}^a = (\partial_\varphi)^a}$$

of a **centre-of-mass unit sphere reference system** $[\int_{\partial\Sigma}\vec{x}\,\mathring{\boldsymbol{\epsilon}} = 0]$.

- The integrands: **more information** than the Bondi energy and angular momentum.
- $\exists$ of ∞ many charges is guaranteed by regularity of these expressions. $[{}_sY^\ell_m]$

- The **algebraic conditions** on $\partial\Sigma$: $\boxed{\mathbf{a}_{(-1)} = const \;\&\; \mathring{\mathbf{K}}_{qq0} = \tfrac{1}{2}\,\boldsymbol{\kappa}_0 \cdot \eth\eth\,\boldsymbol{\kappa}_0^{-2}}$ &
  - $\widehat{\mathbf{N}} \longrightarrow \widehat{\mathbf{N}}_0 + \widehat{\mathbf{N}}_1\,\omega + \widehat{\mathbf{N}}_2\,\omega^2 + \widehat{\mathbf{N}}_3\,\omega^3 + \widehat{\mathbf{N}}_4\,\omega^4 + \omega^4 w_{\widehat{\mathbf{N}}}(\omega)$
  - $\mathbf{K} \longrightarrow \mathbf{K}_0 + \mathbf{K}_1\,\omega + \mathbf{K}_2\,\omega^2 + \mathbf{K}_3\,\omega^3 + \mathbf{K}_4\,\omega^4 + \omega^4 w_{\mathbf{K}}(\omega)$
  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$
- We get a **Fuchsian-type** (singular at $\omega = 0$) **equation** for the vector-valued variable $\underline{W} = (w_{\widehat{\mathbf{N}}}, w_{\mathbf{K}}, w_{\mathbf{k}})^T$, comprised of the residuals, where $p \in \mathscr{S}^2$ and $0 < \omega < \omega_0$:

$$\boxed{\begin{aligned}\partial_\omega \underline{W}(\omega,p) = {}& \frac{1}{\omega} diag(0,-3,-1)\,\underline{W}(\omega,p) \\ & + \underline{H}\left(\omega, p; \widehat{\mathbf{N}}_4(p), \mathbf{K}_1(p), \mathbf{k}_2(p), \underline{W}(\omega,p), \eth\underline{W}, \overline{\eth}\underline{W}, \eth\overline{\eth}\underline{W}\right)\end{aligned}} \qquad (*)$$

- where $\underline{H}$ is a (lengthy, but **explicitly known**) vector-valued function that is **smooth in each of its arguments**, and **regularly extends to** $\omega = 0$.
- A "formal solution" of $(*)$ can be given by this integral equation

$$\boxed{\underline{W}(\omega,p) = diag\left[1, \omega^{-3}, \omega^{-1}\right] \times \int_0^\omega diag\left[1, s^3, s\right] \times \underline{H}(s,p)\,\mathrm{d}s} \qquad (**)$$

- Since the integrand regularly extends to $s = 0$, we can perform the change of variables in the integral by replacing s with the product $\omega \cdot \tau$, which yields: $[0, \omega_0)$

$$\boxed{\tfrac{1}{\omega}\,\underline{W}(\omega,p) = \int_0^1 diag\left[1, \tau^3, \tau\right] \times \underline{H}(\omega\cdot\tau, p)\,\mathrm{d}\tau} \qquad (***)$$

Our second main result:

Theorem II.

- Choose **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{qq})$ on Σ such that it **satisfies the asymptotically hyperboloidal falloff conditions**, i.e., $\omega^2 h_{ij}$ & $\omega \mathring{K}_{ij}$ are **regular** on $\Sigma \cup \partial\Sigma$.
- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth** [i.e., of class $C^\infty\big((0,\omega_0), C^\infty(\mathbb{S}^2)\big)$], solutions on Σ such that $\widehat{\mathbf{N}} > 0$ there.
- Then, the constrained fields $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are also of class $C^\infty\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ on the whole of $\Sigma \cup \partial\Sigma$, i.e., **no logarithmic singularities occur**, **if and only if** the asymptotically hyperboloidal initial data set under consideration **admits well-defined Bondi energy and angular momentum**, and, **in addition**, for the free data

$$\mathbf{a}_{(-1)} = const \quad \& \quad \mathring{\mathbf{K}}_{qq0} = \tfrac{1}{2}\,\boldsymbol{\kappa}_0 \cdot \bar{\eth}\eth\,\boldsymbol{\kappa}_0^{-2}$$

also hold on $\partial\Sigma$.

Summary:

- We proved that **the existence of well-defined Bondi energy and angular momentum**, together with some mild restrictions on the free data, implies that **the generic solutions of the parabolic-hyperbolic form of the constraint equations are free of logarithmic singularities**.
- We eliminated the restrictions imposed by Beyer and Ritchie on constrained fields. Additionally, we considerably weakened their restrictions on free data.
- Since Cauchy developments of smooth, asymptotically hyperboloidal initial data have smooth, conformal boundaries, **our result confirms** the **smoothness assumptions** used by Penrose when he introduced the concept of **asymptotically simple spacetimes**.
- Is there a way to **control the asymptotic charges** when constructing solutions to **the parabolic-hyperbolic form of the constraints**? [Work in progress.]

Thanks for your attention