Hyperboloidal neutron star and black hole in spherical symmetry

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IAC3, University of the Balearic Islands

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Based on 2505.XXXXX [gr-qc]

Bondi accretion

Accretion onto a small black hole at the center of a neutron star

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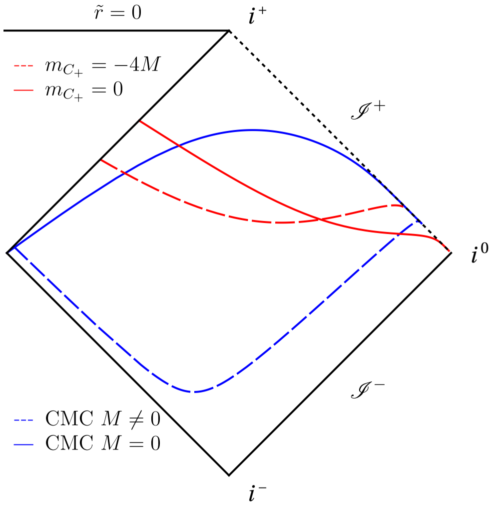
2102.09574 [astro-ph.HE]

Hyperboloidal slice to know mass of system



Peterson et al. *Phys. Rev. D* 110 (2024)

Setup on Cauchy slices

Line element:

$$d\tilde{s}^2 = -e^{\nu(\tilde{r})}d\tilde{t}^2 + e^{\lambda(\tilde{r})}d\tilde{r}^2 + \tilde{r}^2 d\sigma^2$$

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TOV equations (assuming perfect fluid):

$$\partial_{\tilde{r}} m(\tilde{r}) = 4\pi\tilde{r}^2\rho(\tilde{r}) \quad \rightarrow \quad e^{\lambda(\tilde{r})} = \left(1 - \frac{2m(\tilde{r})}{\tilde{r}}\right)^{-1}$$

$$\partial_{\tilde{r}}\nu(\tilde{r}) = 2\left(\frac{m(\tilde{r})}{\tilde{r}^2} + 4\pi\tilde{r}P(\tilde{r})\right)\left(1 - \frac{2m(\tilde{r})}{\tilde{r}}\right)^{-1}$$

$$\partial_{\tilde{r}}P(\tilde{r}) = -\frac{1}{2}\left(P(\tilde{r}) + \rho(\tilde{r})\right)\partial_{\tilde{r}}\nu(\tilde{r})$$

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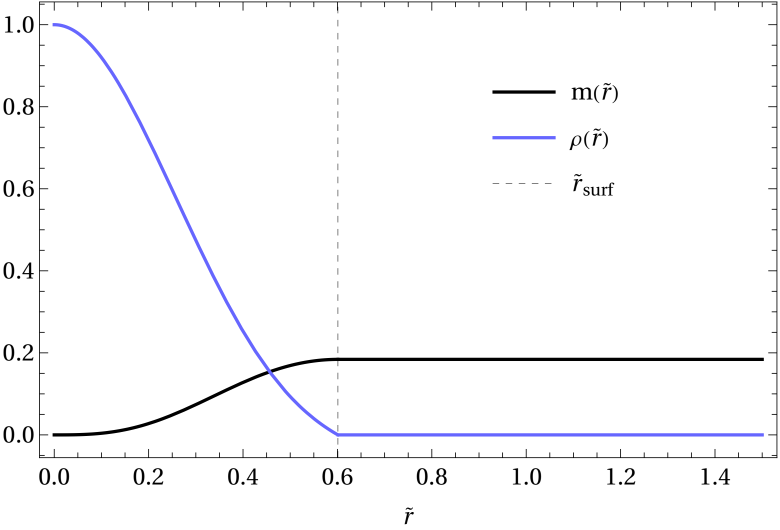
Polytropic equation of state: $P(\tilde{r}) = K[\rho(\tilde{r})]^\Gamma$

Neutron star profiles



Choice of hyperboloidal slice

Hyperboloidal time: $\tilde{t} = t + h(\tilde{r})$

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Constant-mean curvature slices: $\tilde{K} = -\frac{1}{\sqrt{-\tilde{g}}}\partial_a\left(\sqrt{-\tilde{g}}\,\tilde{n}^a\right) \equiv \text{constant}$

$$\downarrow$$

$$\begin{aligned}\frac{e^\nu h'}{\sqrt{e^\lambda - e^\nu (h')^2}} &= -\frac{1}{\tilde{r}^2}\left[\int K_{\text{CMC}}\tilde{r}^2\sqrt{e^\lambda e^\nu}d\tilde{r} + C_{\text{CMC}}\right] \\ &\equiv int(\tilde{r}),\end{aligned}$$

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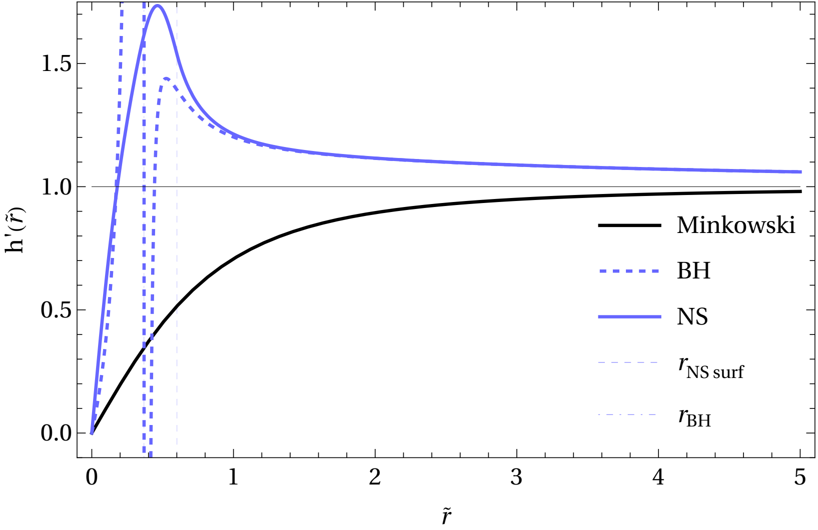
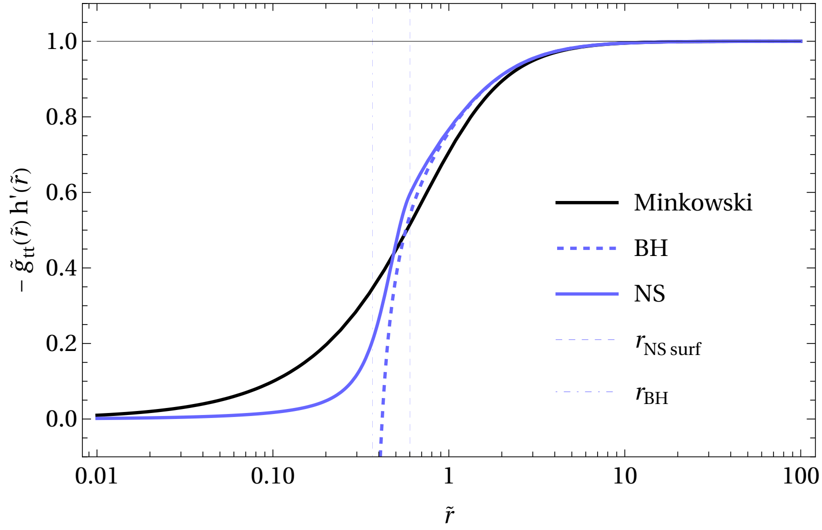
Boost:

$$h'(\tilde{r}) = \pm int(\tilde{r})\sqrt{\frac{e^{\lambda(\tilde{r})}}{e^{\nu(\tilde{r})}\left(e^{\nu(\tilde{r})}+[int(\tilde{r})]^2\right)}},$$

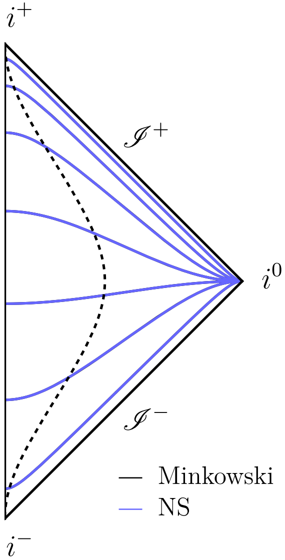
Boost function

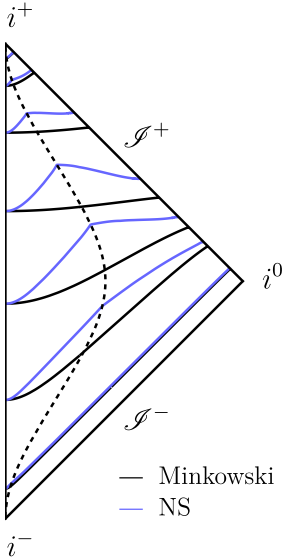


Rescaled boost function



Penrose diagrams





Compactification by imposing conformal flatness

Compactification: $\tilde{r} = \frac{r}{\bar{\Omega}(r)}$

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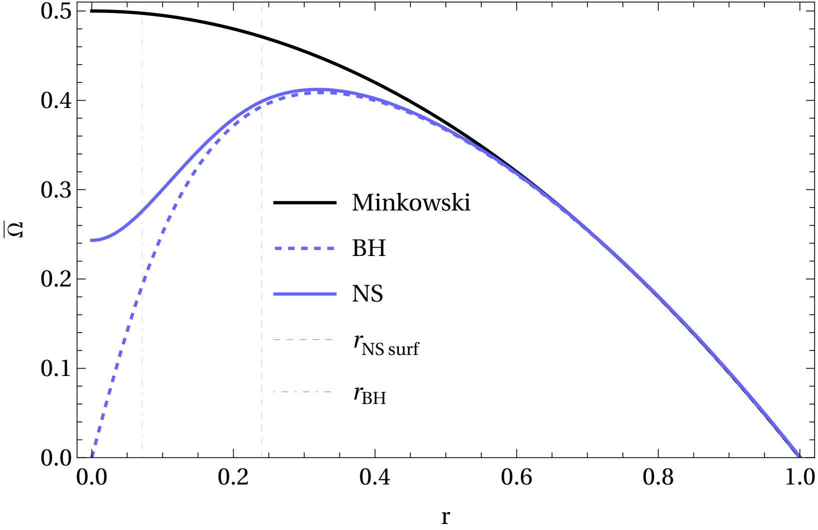
Impose:

$$\gamma_{rr} = \left[\left(1 - \frac{2m(\frac{r}{\bar{\Omega}})\bar{\Omega}}{r}\right)^{-1} - e^{\nu(\frac{r}{\bar{\Omega}})}\left(h'(\frac{r}{\bar{\Omega}})\right)^2\right]\left(\frac{\bar{\Omega} - r\bar{\Omega}'}{\bar{\Omega}}\right)^2 = 1$$

Compactification factors



Procedure

1. Solve TOV for NS on a Cauchy slice
2. Express it in isotropic form
3. Add BH in isotropic form
4. Solve Hamiltonian constraint
5. Hyperboloidalize
6. Compactify

Transforming to isotropic radius

Want to add contributions as

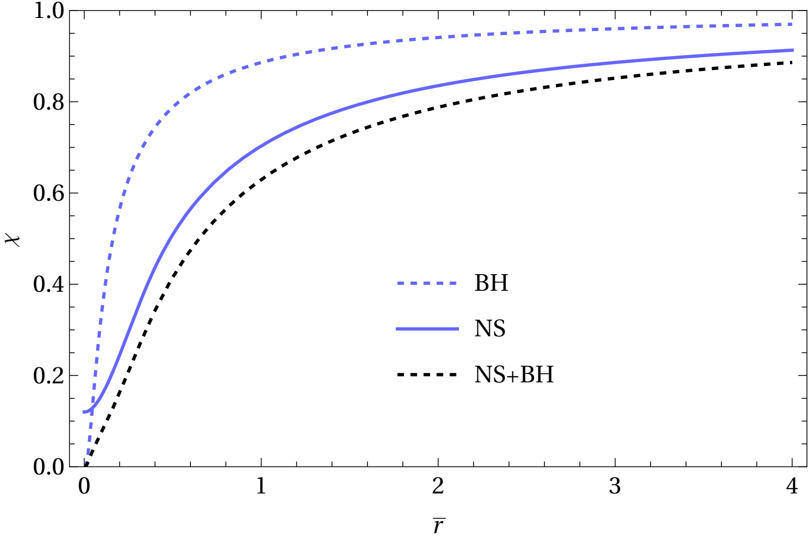
$$\psi = \psi_{\text{NS}} + \psi_{\text{BH}} + \delta\psi$$

with

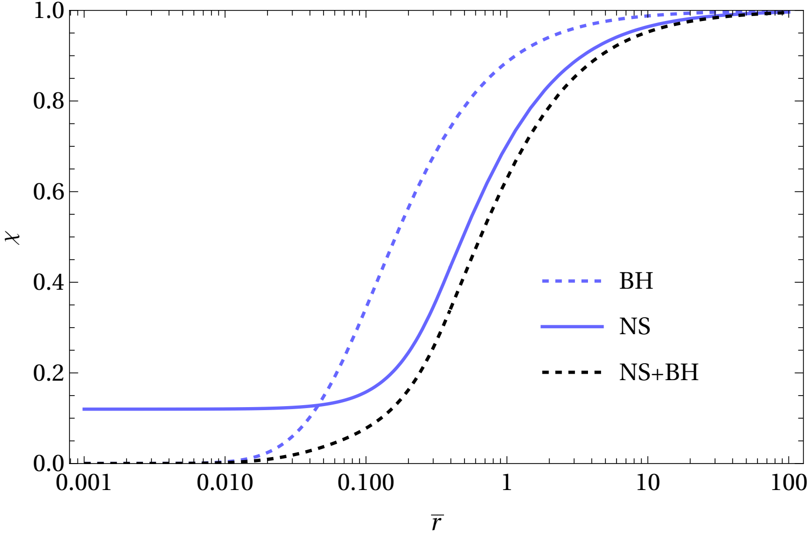
$$d\tilde{s}^2 = -f d\tilde{t}^2 + \psi^4 \left(d\bar{r}^2 + \bar{r}^2 d\sigma^2 \right)$$

- ψ_{NS} determined from numerical solution
- $\psi_{\text{BH}} = \frac{m_{\text{BH}}}{2\bar{r}}$
- $\delta\psi$ to solve the Hamiltonian constraint for

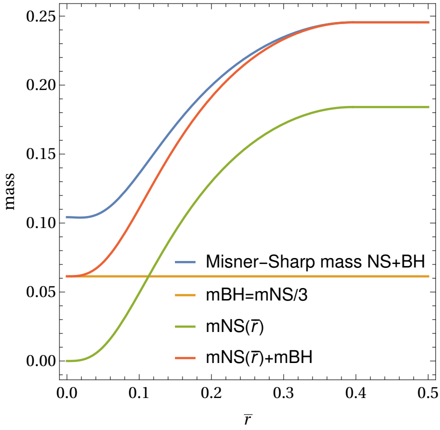
Addition of conformal factors

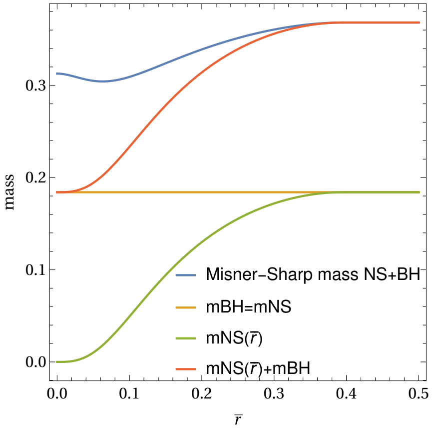


Addition of conformal factors (logscale)



Misner-Sharp mass





Left: $m_{\rm BH} = m_{\rm NS}/3$, right: $m_{\rm BH} = m_{\rm NS}$.

Setup

Following [3]

$$\Delta\psi = -2\pi\psi^5\rho \quad \text{using} \quad \rho = \psi^m\bar{\rho}$$

Set

$$\Delta\psi_{\rm NS} = -2\pi\psi_{\rm NS}^5\rho_{\rm NS}, \qquad \Delta\psi_{\rm BH} = 0, \qquad \bar{\rho} = \psi_{\rm NS}^{-m}\rho_{\rm NS}$$

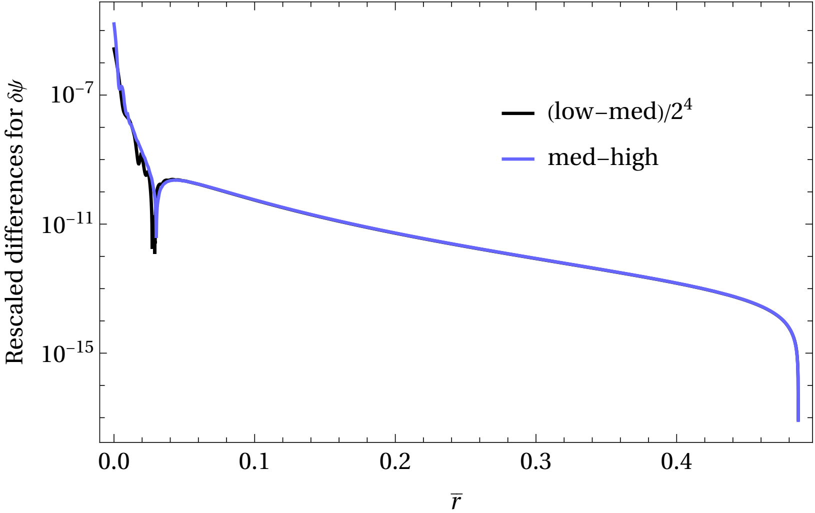
Solve with $m = -6$

$$\Delta\delta\psi = 2\pi\rho_{\rm NS}\left(\psi_{\rm NS}^5 - \frac{(\psi_{\rm NS}+\psi_{\rm BH}+\delta\psi)^{5+m}}{\psi_{\rm NS}^m}\right),$$

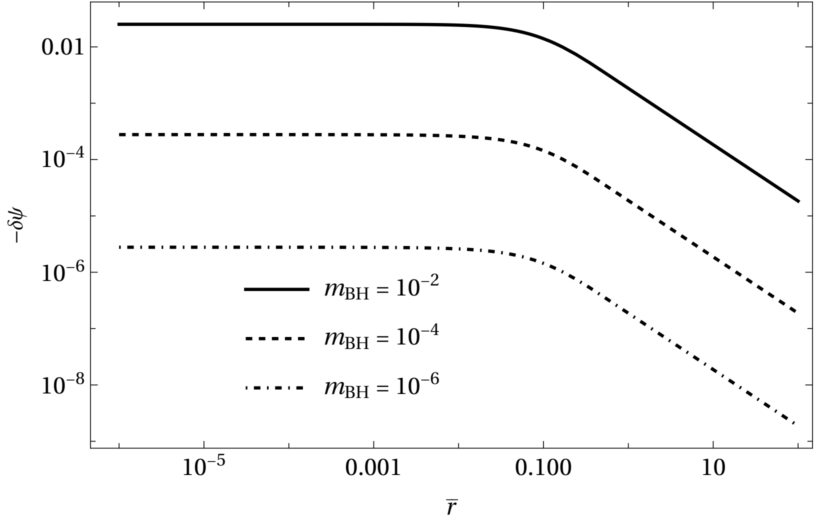
$$\partial_{\bar{r}}^2\delta\psi + \frac{2}{\bar{r}}\partial_{\bar{r}}\delta\psi = 2\pi\rho_{\rm NS}\left(\psi_{\rm NS}^5 - \frac{\psi_{\rm NS}^6}{(\psi_{\rm NS}+\psi_{\rm BH}+\delta\psi)}\right),$$

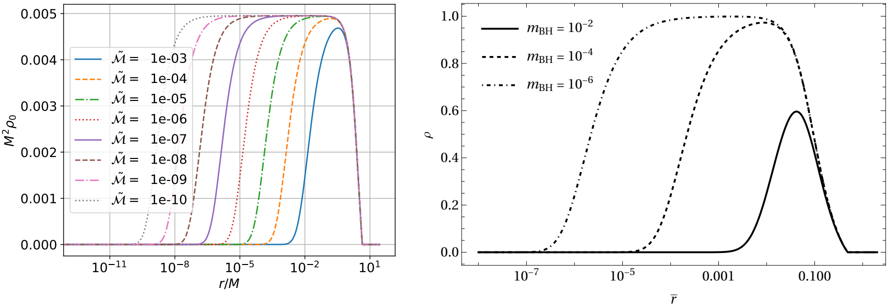
assuming $\delta\psi \sim A/\bar{r}$ at NS's surface.

Convergence



$\delta\psi$ solutions for small BH





Left: Richards, Baumgarte and Shapiro. *Phys. Rev. D* 103.10 (2021), right: this work.

Radial transformation

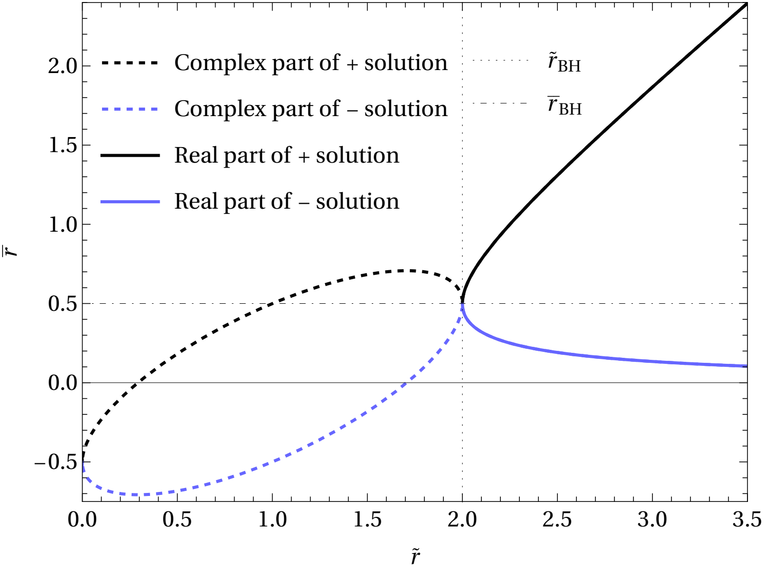
$$d\tilde{l}^2 = \psi^4 \left(d\bar{r}^2 + \bar{r}^2 d\sigma^2 \right) = g_{\tilde{r}\tilde{r}}\, d\tilde{r}^2 + \tilde{r}^2 d\sigma^2,$$

Radial transformation

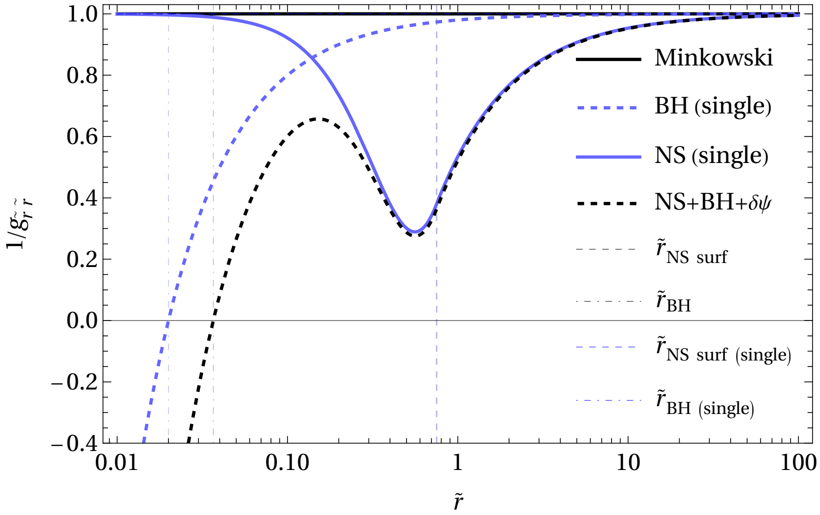
$$d\tilde{l}^2 = \psi^4 \left(d\bar{r}^2 + \bar{r}^2 d\sigma^2\right) = g_{\tilde{r}\tilde{r}}\, d\tilde{r}^2 + \tilde{r}^2 d\sigma^2,$$

Difficulty: $g_{\tilde{r}\tilde{r}}$ changes sign at the horizon.

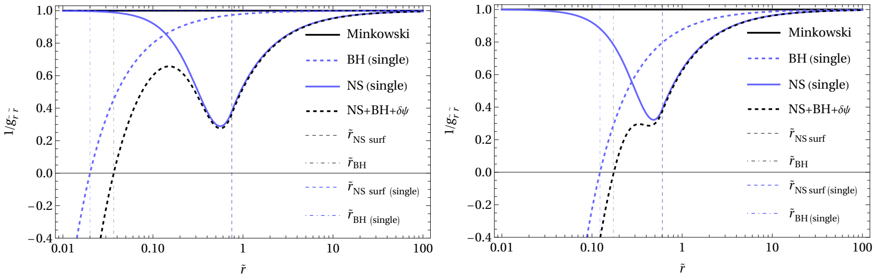
Outside/inside of horizon of BH – $\tilde{r} = \left(1 + \frac{m_{BH}}{2\,\bar{r}}\right)^2 \bar{r}$



Metric deformation



Comparison of effect of BH masses



Left: $m_{\rm BH} = 10^{-2}$, right: $m_{\rm BH} = m_{\rm NS}/3$.

Determine boost

Choose

$$g_{\tilde{t}\tilde{t}} \doteq -\frac{1}{g_{\tilde{r}\tilde{r}}} \equiv -f(\tilde{r})$$

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Introduce in

$$h'(\tilde{r}) = -\frac{\left(\frac{K_{\mathrm{CMC}}\tilde{r}}{3} + \frac{C_{\mathrm{CMC}}}{\tilde{r}^3}\right)}{f(\tilde{r})\sqrt{f(\tilde{r}) + \left(\frac{K_{\mathrm{CMC}}\tilde{r}}{3} + \frac{C_{\mathrm{CMC}}}{\tilde{r}^3}\right)^2}}$$

Determine boost

Choose

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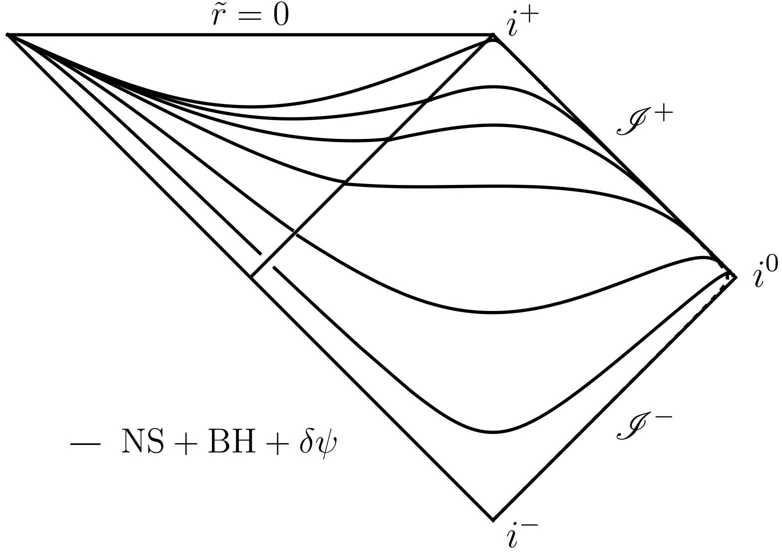
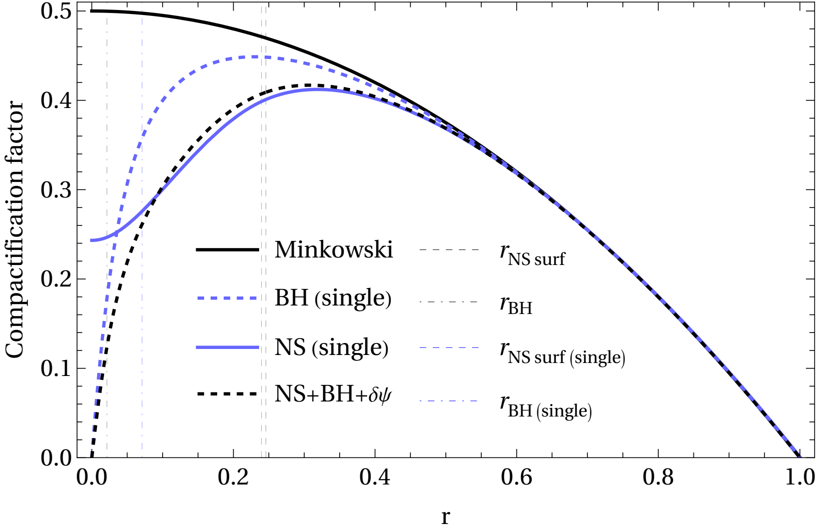
Tune value of C_{CMC} for boost to diverge at trumpet.

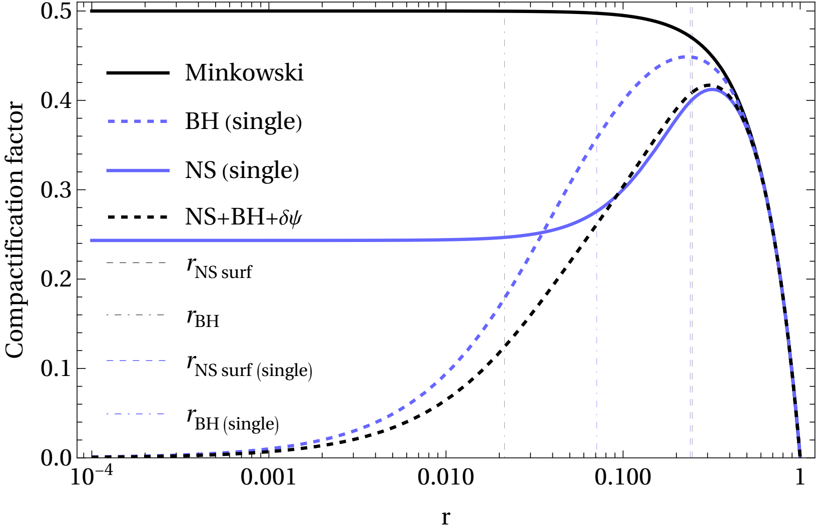
Penrose diagram



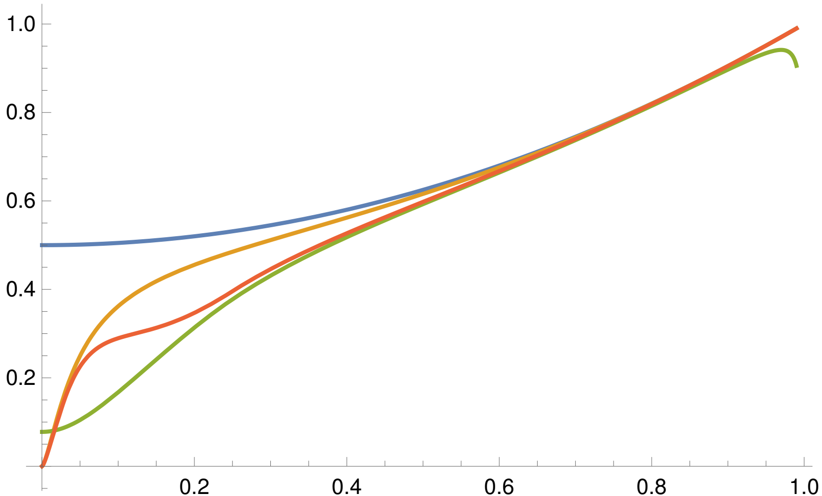
Compactification factor



Compactification factor – log scale



Ready as initial data for evolutions



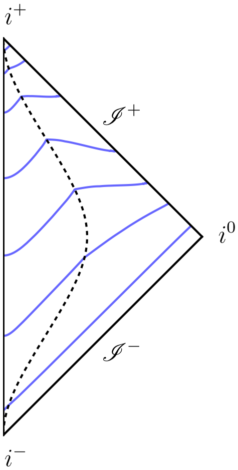
Future plans

In spherical symmetry:

- Evolve NS initial data (Einstein + relativistic Euler)
- Evolve perturbed NS
- Bondi accretion: evolve NS + small BH initial data

Beyond spherical symmetry:

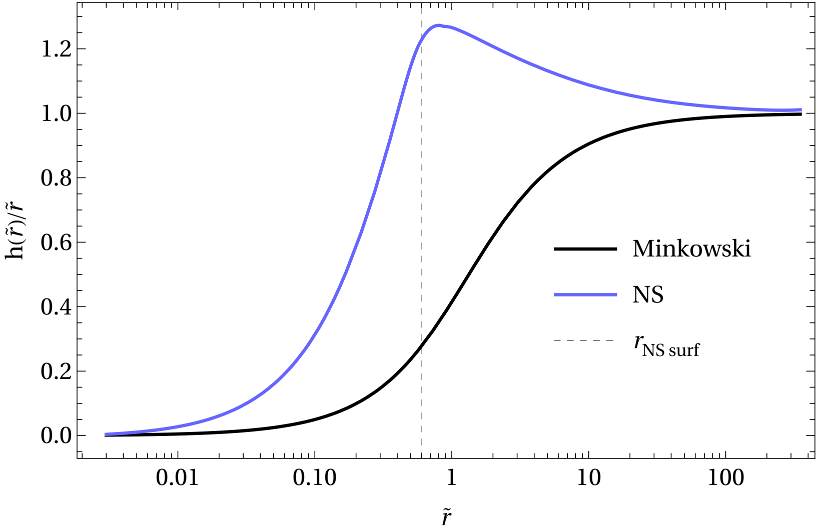
- Hyperboloidalize superposition of bodies in different locations
- Evolve 3D perturbed NS
- ...



Thanks for listening! Questions?

Backup slides

Integration of the height function



Tortoise-like coordinate

Express metric as: $d\tilde{s}^2 = \Xi^2 \left(-d\tilde{t}^2 + d\tilde{r}_*^2\right) \equiv -\Xi^2 \, d\tilde{u} \, d\tilde{v}$

For NS: $d\tilde{s}^2 = e^{\nu(\tilde{r})} \left(-d\tilde{t}^2 + d\tilde{r}_*^2\right)$ with $d\tilde{r}_* = \sqrt{\frac{e^{\lambda(\tilde{r})}}{e^{\nu(\tilde{r})}}} d\tilde{r}$

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Usual compactification along null directions:

$$\begin{aligned} \tilde{U} &= \tilde{t} - \tilde{r}_*, & \tilde{V} &= \tilde{t} + \tilde{r}_*, \\ U &= \arctan \tilde{U}, & V &= \arctan \tilde{V}, \\ T &= \frac{V+U}{2}, & R &= \frac{V-U}{2}. \end{aligned}$$

Tortoise-like coordinate $d\tilde{r}_* = \frac{1}{f(\tilde{r})} d\tilde{r}$