Evolution of Energy and Linear Momentum in Asymptotically Hyperboloidal Initial Data Sets

Saradha Senthil Velu

Virtual Infinity Seminar

April 11, 2025

University of Tübingen

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]
- Conceptually distinct from the *Hamiltonian* formulation

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]
- Conceptually distinct from the *Hamiltonian* formulation
- Approach: use Einstein evolution equations on a hyperboloidal initial data set

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]
- Conceptually distinct from the *Hamiltonian* formulation
- Approach: use Einstein evolution equations on a hyperboloidal initial data set
  - Construct an appropriate time function

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]
- Conceptually distinct from the *Hamiltonian* formulation
- Approach: use Einstein evolution equations on a hyperboloidal initial data set
  - Construct an appropriate time function
  - Build a hyperboloidal foliation

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]
- Conceptually distinct from the *Hamiltonian* formulation
- Approach: use Einstein evolution equations on a hyperboloidal initial data set
    - Construct an appropriate time function
    - Build a hyperboloidal foliation
- Derive a **mass loss formula** analogous to the Bondi formula[2]

---

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Outline

- **Energy** and **linear momentum** as geometric invariants on asymptotically hyperboloidal initial data sets [1]
- Conceptually distinct from the *Hamiltonian* formulation
- Approach: use Einstein evolution equations on a hyperboloidal initial data set
    - Construct an appropriate time function
    - Build a hyperboloidal foliation
- Derive a **mass loss formula** analogous to the Bondi formula[2]
- Related: energy loss using Liu-Yau quasi-local mass[3]

[1] Michel, B.: Geometric invariance of mass-like asymptotic invariants (2011)

[2] Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity VII (1962)

[3] Chen, P.-N., Wang, M.-T., Yau, S.-T.: Conserved quantities on asymptotically hyperbolic initial data sets (2014)

Introduction

Consider a spacetime (L, γ) and a time function $t : \mathbb{R} \supseteq (a, b) \to \mathbb{R}$.

- Level sets of t given by M_t with induced metric g and extrinsic curvature K produce a local foliation of (L, γ).
- The constraints operator is defined by:

$$\Phi(g, K) := \begin{pmatrix} \mathrm{R} + (\mathrm{tr}_g K)^2 - |K|_g^2 \\ 2\mathrm{div}_g(K - \mathrm{tr}_g K \cdot g) \end{pmatrix} =: \begin{pmatrix} \Phi_H \\ \Phi_M \end{pmatrix},$$

where $\Phi : \mathcal{M} \times_M S^2(T^*M) \to \mathcal{C}^\infty(M) \times \Gamma(T^*M)$, with $\mathcal{M}$ the bundle of metrics, and $S^2(T^*M)$ the bundle of symmetric $(0, 2)$-tensors on M.

- The Einstein Constraint Equations require that:

$$\Phi\left({}^{(t)}g, {}^{(t)}K\right) = (\rho, J).$$

- The vacuum Einstein Evolution Equations are given by:

$$\begin{cases} \mathcal{L}_\nu\, {}^{(t)}g = 2\, {}^{(t)}K, \\ \mathcal{L}_\nu {}^{(t)}K - \dfrac{{}^{(t)}\nabla^2 N}{N} = 2\left({}^{(t)}K \circ {}^{(t)}K\right) - {}^{(t)}\mathrm{Ric} - \left({}^{(t)}\operatorname{tr}{}^{(t)}K\right){}^{(t)}K, \end{cases}$$

where $\nu = \frac{1}{N}\left(\partial_t - X\right)$, with $N \in C^\infty(M)$ and $X \in \Gamma(TM)$.

Hyperboloids in Minkowski

We denote by $(\mathbb{H}^3, b, b)$ the hyperboloid of radius 1 embedded in Minkowski as a totally umbilic spacelike hypersurface

$$\mathbb{H}^3 := \{(t, x) \in \mathbb{R}^{3,1} \mid t^2 = r^2 + 1\},$$
$$b = \frac{1}{1+r^2} dr^2 + r^2 \sigma_{\alpha\beta} du^\alpha du^\beta,$$

where σ is the standard round metric on the unit sphere. On the hyperboloid, we use coordinates $x = (r, u^J)$, where $u^J \in \mathbb{S}^2$ denote the standard polar coordinates on the sphere.

Benoit Michel Charges on Asymptotically Hyperboloidal Initial Data Sets

Consider an Initial Data Set (M, g, K) and a diffeomorphism at infinity Ψ providing coordinates in which the above is asymptotic to the standard hyperboloid in Minkowski. We denote the errors as follows,

$$\dot{g} = \Psi^* g - b, \text{ and } \dot{K} := \Psi^* K - b.$$

- We use the constraint operator Φ as the charge density.
- Note that $\Phi_b = (0, 0)$.

The linearization of Φ at $(\mathbb{H}^3, b, b)$ is given by

$$D\Phi_b(\dot{g}, \dot{K}) = \begin{pmatrix} \mathrm{div}_b\mathrm{div}_b\dot{g} + \Delta_b\mathrm{tr}_b\dot{g} - 2\mathrm{tr}_b\dot{g} + 4\mathrm{tr}_b\dot{K} \\ d\mathrm{tr}_b\dot{g} - b^{ij}\nabla\dot{g}_{ij} - 2(\mathrm{div}_b\dot{g} - d\mathrm{tr}_b\dot{g}) + 2\mathrm{div}_b\dot{K} - d\mathrm{tr}_b\dot{K} \end{pmatrix}.$$

Taylor expanding Φ and contracting with a test function $\mathcal{V} \in \Gamma(\mathbb{R} \oplus T^*\mathbb{H})$ leads to

$$\langle \Phi(g, K) - \Phi_b, \mathcal{V} \rangle_b = \langle D\Phi_b(\dot{g}, \dot{K}), \mathcal{V} \rangle_b + \mathcal{Q}(\mathcal{V}, (\dot{g}, \dot{K})).$$

Now, integrating by parts:

$$\langle \Phi(g, K), \mathcal{V} \rangle_b = \mathrm{div}_b\mathbb{U}(\mathcal{V}, (\dot{g}, \dot{K})) + \langle D^*\Phi_b\mathcal{V}, (\dot{g}, \dot{K}) \rangle_b + \mathcal{Q}(\mathcal{V}, (\dot{g}, \dot{K})).$$

With the above, we have a *geometric invariant* on (M, g, K) for each non-trivial test function $\mathcal{V}$ for which $D^*\Phi_b\mathcal{V} = 0$, given by the following,

$$m(\dot{g}, \dot{K}, \mathcal{V}) := \lim_{k\to\infty} \frac{1}{16\pi} \oint_{\mathcal{S}_k} \mathbb{U}(\mathcal{V}, (\dot{g}, \dot{K}))(\nu) dS,$$

where $S_k = \partial B_k$ is smooth and compact, $(B_k)_{k\in\mathbb{N}}$ is a non-decreasing exhaustion of the background manifold, and ν and dS are the outer unit normal and surface measure on S_k with respect to b.

- We obtained geometric invariants from elements $\mathcal{V}$ of $ker(D^*\Phi)$.

[4] Maerten, D.: Killing initial data revisited (2004)

[5] Moncrief V.: Spacetime symmetries and linearization stability of the einstein equations (1974)

- We obtained geometric invariants from elements $\mathcal{V}$ of $ker(D^*\Phi)$.
- These are the Killing Initial Data (KIDs) of $(\mathbb{H}^3, b, b)$ [4] [5]

[4] Maerten, D.: Killing initial data revisited (2004)

[5] Moncrief V.: Spacetime symmetries and linearization stability of the einstein equations (1974)

- We obtained geometric invariants from elements $\mathcal{V}$ of $ker(D^*\Phi)$.
- These are the Killing Initial Data (KIDs) of $(\mathbb{H}^3, b, b)$ [4,5]
- The KID corresponding to time translation $\mathcal{V}_0$ and the KIDs corresponding to the spatial translation $\mathcal{V}_i$ lead to the conserved quantities of energy and linear momenta respectively.

[4] Maerten, D.: Killing initial data revisited (2004)

[5] Moncrief V.: Spacetime symmetries and linearization stability of the einstein equations (1974)

Writing $\mathbb{U}(\mathcal{V}, (\dot{g}, \dot{K}))$ explicitly for a KID $\mathcal{V} = (V, Y) = (V, \alpha^{\#})$, we have

$$m(\dot{g}, \dot{K}, \mathcal{V}) = \lim_{r\to\infty} \frac{1}{16\pi} \oint \Big[V\big(\mathrm{div}_b \dot{g} - d\mathrm{tr}_b \dot{g}\big) - \iota_{\nabla V}\dot{g} + (\mathrm{tr}_b \dot{g}) dV$$
$$+ 2\big(\iota_\alpha \dot{K} - (\mathrm{tr}_b \dot{K})\alpha)\big) + (\mathrm{tr}_b \dot{g})\iota_\alpha K_{\mathbb{H}^3}$$
$$+ \langle K_{\mathbb{H}^3}, \dot{g}\rangle_b \alpha - 2\iota_\alpha(\dot{g} \circ K_{\mathbb{H}^3})\Big] \, (n) \, dA_{\mathbb{S}^2_r}.$$

We note that

$$n = \frac{\partial_r^{\mathrm{tang}}}{\big|\partial_r^{\mathrm{tang}}\big|_\eta} = -r\partial_{t|_{\mathbb{H}^3}} + \sqrt{1+r^2}\,\partial_{r|_{\mathbb{H}^3}}.$$

We obtain the definition of energy of an asymptotically hyperboloidal initial data set:

Energy

$$E = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2_r} V^2\big(\mathrm{div}_b\, \dot{g} - d\,\mathrm{tr}_b \dot{g}\big) - V \mathrm{tr}_\sigma(\dot{g}_{\alpha\beta}) + 2V\mathrm{tr}_\sigma(\dot{K}_{\alpha\beta})\, dA_{\mathbb{S}^2_r},$$

for $\mathcal{V}_0 = \{V = \sqrt{1+r^2}, \quad Y = -r\sqrt{1+r^2}\partial_r^{\mathrm{tang}}\}$, We obtain the definition of energy of an asymptotically hyperboloidal initial data set:

Energy

$$E = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2_r} V^2 \left(\mathrm{div}_b\, \dot{g} - d\, \mathrm{tr}_b \dot{g}\right) - V \mathrm{tr}_\sigma(\dot{g}_{\alpha\beta}) + 2V \mathrm{tr}_\sigma(\dot{K}_{\alpha\beta})\, dA_{\mathbb{S}^2_r},$$

for $\mathcal{V}_0 = \{V = \sqrt{1+r^2}, \quad Y = -r\sqrt{1+r^2}\partial_r^{\mathrm{tang}}\}$,

Linear Momentum

$$P^i = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2_r} \left(V^2 \left(\mathrm{div}_b \dot{g} - d \mathrm{tr}_b \dot{g}\right) - V \mathrm{tr}_\sigma(\dot{g}_{\alpha\beta}) + 2V \mathrm{tr}_\sigma(\dot{K}_{\alpha\beta})\right) x^i dA_{\mathbb{S}^2_r},$$

for

$$\begin{aligned}
\mathcal{V}_1 &= \{V_1 = -r\sin\theta\cos\varphi, \quad Y_1 = -\mathrm{grad}_b V_1\},\\
\mathcal{V}_2 &= \{V_2 = -r\sin\theta\sin\varphi, \quad Y_2 = -\mathrm{grad}_b V_2\},\\
\mathcal{V}_3 &= \{V_3 = -r\cos\theta, \quad Y_3 = -\mathrm{grad}_b V_3\}.
\end{aligned}$$

Asymptotically Hyperboloidal IDS

An initial data set (M, g, K, ρ, J) is *asymptotically hyperboloidal* if there exist compact sets $B \subset M$ and $B_0 \subset \mathbb{H}^3$, and a diffeomorphism at infinity

$$\Psi : M \setminus B \to \mathbb{H}^3 \setminus B_0,$$

such that the $(0,2)$-symmetric tensors on $\mathbb{H}^3 \setminus B_0$

$$\dot{g} := \Psi^* g - b, \quad p := \Psi^* K - \Psi^* g$$

have the following asymptotic behavior as $r \to \infty$:

$$\dot{g}_{rr} = \frac{\mathbf{m}_{rr}}{r^5} + \frac{\tilde{\mathbf{m}}_{rr}}{r^6} + O_2(r^{-7}), \qquad p_{rr} = \frac{\mathbf{p}_{rr}}{r^5} + \frac{\tilde{\mathbf{p}}_{rr}}{r^6} + O_1(r^{-7}),$$

$$\dot{g}_{r\alpha} = \frac{\mathbf{g}_{r\alpha}}{r^3} + O_2(r^{-4}), \qquad p_{r\alpha} = \frac{\mathbf{p}_{r\alpha}}{r^3} + O_1(r^{-4}),$$

$$\dot{g}_{\alpha\beta} = {}^0\mathbf{g}_{\alpha\beta} + \frac{{}^g\mathbf{m}_{\alpha\beta}}{r} + \frac{{}^g\tilde{\mathbf{m}}_{\alpha\beta}}{r^2} + O_2(r^{-3}), \quad p_{\alpha\beta} = {}^0\mathbf{p}_{\alpha\beta} + \frac{{}^k\mathbf{m}_{\alpha\beta}}{r} + \frac{{}^k\tilde{\mathbf{m}}_{\alpha\beta}}{r^2} + O_1(r^{-3}).$$

- $\mathbf{m}_{rr}$, $\mathbf{p}_{rr}$, $\tilde{\mathbf{m}}_{rr}$ and $\tilde{\mathbf{p}}_{rr}$ are functions on $\mathbb{S}^2$ that do not depend on r.

- $\mathbf{m}_{rr}$, $\mathbf{p}_{rr}$, $\tilde{\mathbf{m}}_{rr}$ and $\tilde{\mathbf{p}}_{rr}$ are functions on $\mathbb{S}^2$ that do not depend on r.
- $\mathbf{g}_{r\alpha}$ and $\mathbf{p}_{r\alpha}$ are one-forms that do not depend on r.

- $\mathbf{m}_{rr}$, $\mathbf{p}_{rr}$, $\tilde{\mathbf{m}}_{rr}$ and $\tilde{\mathbf{p}}_{rr}$ are functions on $\mathbb{S}^2$ that do not depend on r.
- $\mathbf{g}_{r\alpha}$ and $\mathbf{p}_{r\alpha}$ are one-forms that do not depend on r.
- ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\mathbf{m}_{\alpha\beta}$, ${}^k\mathbf{m}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are symmetric $(0,2)$-tensors on $\mathbb{S}^2$ which do not depend on r.

- $\mathbf{m}_{rr}$, $\mathbf{p}_{rr}$, $\tilde{\mathbf{m}}_{rr}$ and $\tilde{\mathbf{p}}_{rr}$ are functions on $\mathbb{S}^2$ that do not depend on r.
- $\mathbf{g}_{r\alpha}$ and $\mathbf{p}_{r\alpha}$ are one-forms that do not depend on r.
- ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\mathbf{m}_{\alpha\beta}$, ${}^k\mathbf{m}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are symmetric $(0,2)$-tensors on $\mathbb{S}^2$ which do not depend on r.
- we assume that ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are traceless with respect to σ.

- $\mathbf{m}_{rr}$, $\mathbf{p}_{rr}$, $\tilde{\mathbf{m}}_{rr}$ and $\tilde{\mathbf{p}}_{rr}$ are functions on $\mathbb{S}^2$ that do not depend on r.
- $\mathbf{g}_{r\alpha}$ and $\mathbf{p}_{r\alpha}$ are one-forms that do not depend on r.
- ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\mathbf{m}_{\alpha\beta}$, ${}^k\mathbf{m}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are symmetric $(0,2)$-tensors on $\mathbb{S}^2$ which do not depend on r.
- we assume that ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are traceless with respect to σ.
- the indices on these tensors are raised and lowered with respect to $\sigma_{\alpha\beta}$.

- $\mathbf{m}_{rr}$, $\mathbf{p}_{rr}$, $\tilde{\mathbf{m}}_{rr}$ and $\tilde{\mathbf{p}}_{rr}$ are functions on $\mathbb{S}^2$ that do not depend on r.
- $\mathbf{g}_{r\alpha}$ and $\mathbf{p}_{r\alpha}$ are one-forms that do not depend on r.
- ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\mathbf{m}_{\alpha\beta}$, ${}^k\mathbf{m}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are symmetric $(0,2)$-tensors on $\mathbb{S}^2$ which do not depend on r.
- we assume that ${}^0\mathbf{g}_{\alpha\beta}$, ${}^0\mathbf{p}_{\alpha\beta}$, ${}^g\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^k\tilde{\mathbf{m}}_{\alpha\beta}$ are traceless with respect to σ.
- the indices on these tensors are raised and lowered with respect to $\sigma_{\alpha\beta}$.
- All terms in the expansion of $\dot{g}$ are in $C^2(\mathbb{S}^2)$ and those in the expansion of p are in $C^1(\mathbb{S}^2)$.

Comparison to other definitions of asymptotically hyperboloidal initial data sets:

1. Chen, Wang and Yau [6]: no traceless assumption on ${}^{g}\tilde{\mathbf{m}}_{\alpha\beta}$ and ${}^{k}\tilde{\mathbf{m}}_{\alpha\beta}$.
2. Wang [7]: used in the proof of the positive mass theorem with the Jang equation, does not incorporate radiation.

[6] Conserved quantities on asymptotically hyperbolic initial data sets (2014)

[7] The Mass of Asymptotically Hyperbolic Manifolds (2001)

The energy and linear momentum of an asymptotically hyperboloidal initial data set (M, g, K) are given as follows:

Energy of an AH IDS

$$E = \frac{1}{16\pi} \int_{\mathbb{S}^2} 2\mathbf{m}_{rr} + 3\,\mathrm{tr}_\sigma{}^g\mathbf{m} + 2\mathrm{tr}_\sigma{}^k\mathbf{m}\; dA_{\mathbb{S}^2}.$$

Linear Momentum of an AH IDS

$$P^i = \frac{1}{16\pi} \int_{\mathbb{S}^2} \left(2\mathbf{m}_{rr} + 3\,\mathrm{tr}^g_\sigma\mathbf{m} + 2\mathrm{tr}^k_\sigma\mathbf{m}\right) x^i\, dA_{\mathbb{S}^2},$$

where $x^i,\ i = 1, 2, 3$ denote the first spherical harmonics on the unit sphere $\mathbb{S}^2$.

Choice of Evolution

- The vector field ν chosen for evolution is everywhere timelike - definition of a local time function

[8] Zenginoğlu, A.: Bridging time across null horizons (2025)

Choice of Evolution

- The vector field ν chosen for evolution is everywhere timelike - definition of a local time function
- We need a time function providing a suitable foliation by hyperboloids at infinity.

[8] Zenginoğlu, A.: Bridging time across null horizons (2025)

Choice of Evolution

- The vector field ν chosen for evolution is everywhere timelike - definition of a local time function
- We need a time function providing a suitable foliation by hyperboloids at infinity.
- Other foliations: foliations by asymptotically Euclidean slices or Milne foliations do *not* lead to a measurement of radiation[8].

[8] Zenginoğlu, A.: Bridging time across null horizons (2025)

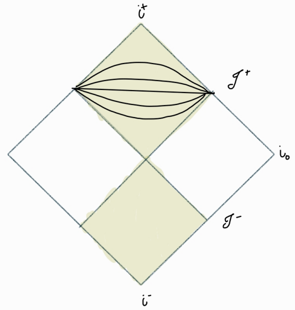

Figure 1: Milne foliation

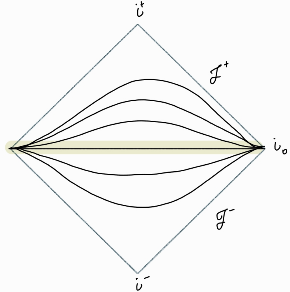

Figure 2: foliations by $t =$ constant slices (AE)

Consider the Minkowski spacetime $(\mathbb{R}^{3,1}, \eta)$, and a time function

$$\tau := t - h(r),$$

where the *height function* $h(r)$ needs to be chosen appropriately to ensure that $\nabla\tau$ is timelike near infinity.

The $(3+1)$ decomposition of Minkowski with respect to τ is:

$$ds^2 = -d\tau^2 - 2h'(r)d\tau dr + (1 - (h'(r))^2)dr^2 + r^2 d\sigma^2.$$

For the choice

$$h'(r) = 1 - \frac{1}{2r^2} + O_2(r^{-3}),$$

we obtain that the $\{\tau = \text{const}\}$ hypersurfaces are hyperboloids of radius 1. The standard hyperboloidal initial data $(\mathbb{H}^3, b, b)$ is obtained with the choice

$$h(r) = \sqrt{1 + r^2},$$

and the lapse and shift on this initial data corresponding to the standard Killing time in Minkowski are

$$N = \sqrt{1 + r^2}\,, \qquad X^r = -r\sqrt{1 + r^2}.$$

It is easy to check that:

$$\mathcal{L}_{\partial_\tau} b = \mathcal{L}_{N\nu + X} = 0.$$

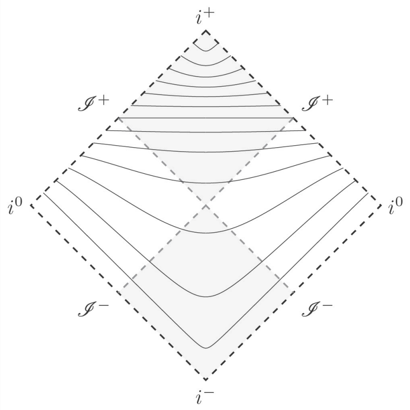

Figure 3: Hyperboloidal foliation [9]

[9] Zenginoğlu, A.: How to draw Penrose diagrams

Evolution of Energy and Linear Momentum

Theorem

Let (M, g, K, ρ, J) be an asymptotically hyperboloidal initial data set with respect to a diffeomorphism $\Psi : \mathbb{H}^3 \setminus B_0 \to M \setminus B$, where $B \subset M$, $B_0 \subset \mathbb{H}^3$ are compact sets, and assume that $\mathrm{supp}(\rho), \mathrm{supp}(J) \subset B$.

Evolution of Energy and Linear Momentum

Theorem

Let (M, g, K, ρ, J) be an asymptotically hyperboloidal initial data set with respect to a diffeomorphism $\Psi : \mathbb{H}^3 \setminus B_0 \to M \setminus B$, where $B \subset M$, $B_0 \subset \mathbb{H}^3$ are compact sets, and assume that $\mathrm{supp}(\rho), \mathrm{supp}(J) \subset B$. Let τ be the standard time function defining a foliation of standard hyperboloids in Minkowski, and let N and X be the corresponding lapse and shift.

Evolution of Energy and Linear Momentum

Theorem

Let (M, g, K, ρ, J) be an asymptotically hyperboloidal initial data set with respect to a diffeomorphism $\Psi : \mathbb{H}^3 \setminus B_0 \to M \setminus B$, where $B \subset M$, $B_0 \subset \mathbb{H}^3$ are compact sets, and assume that $\mathrm{supp}(\rho), \mathrm{supp}(J) \subset B$. Let τ be the standard time function defining a foliation of standard hyperboloids in Minkowski, and let N and X be the corresponding lapse and shift. We denote the foliation of the globally hyperbolic development of $(M, g, K) = \{\tau = 0\}$, by

$$(M \times \{\tau\}, g(\tau), K(\tau)),$$

with $\tau \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$.

Evolution of Energy and Linear Momentum

Theorem

Let (M, g, K, ρ, J) be an asymptotically hyperboloidal initial data set with respect to a diffeomorphism $\Psi : \mathbb{H}^3 \setminus B_0 \to M \setminus B$, where $B \subset M$, $B_0 \subset \mathbb{H}^3$ are compact sets, and assume that $\mathrm{supp}(\rho), \mathrm{supp}(J) \subset B$. Let τ be the standard time function defining a foliation of standard hyperboloids in Minkowski, and let N and X be the corresponding lapse and shift. We denote the foliation of the globally hyperbolic development of $(M, g, K) = \{\tau = 0\}$, by

$$(M \times \{\tau\}, g(\tau), K(\tau)),$$

with $\tau \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. Then, the energy $E(\tau)$ is well-defined on each leaf of the foliation $(M \times \{\tau\}, g(\tau), K(\tau))$ and satisfies

$$\frac{\partial}{\partial \tau} E(\tau) \;=\; -\frac{1}{4\pi} \int_{\mathbb{S}^2} \left| {}^0\mathbf{g} + {}^0\mathbf{p} \right|_\sigma^2 dA_{\mathbb{S}^2}.$$

Sketch of Proof: The Einstein evolution equations in vacuum are given by

$$\frac{\partial}{\partial \tau} g = 2NK + \mathcal{L}_X g,$$

and

$$\frac{\partial}{\partial \tau} K = \mathrm{Hess} N - N \mathrm{Ric} + 2N(K \circ K) - NHK + \mathcal{L}_X K.$$

Sketch of Proof: The Einstein evolution equations in vacuum are given by

$$\frac{\partial}{\partial \tau} g = 2NK + \mathcal{L}_X g,$$

and

$$\frac{\partial}{\partial \tau} K = \mathrm{Hess} N - N\mathrm{Ric} + 2N(K \circ K) - NHK + \mathcal{L}_X K.$$

Using the above, we get:

$$\begin{aligned}\frac{\partial}{\partial \tau} E(\tau) = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2_r} \Big[& V^2 \mathrm{div}_b (2NK + \mathcal{L}_X g)(\partial_r) \\ & - V^2 d\mathrm{tr}_b (2NK + \mathcal{L}_X g)(\partial_r) - V\mathrm{tr}_b (2NK_{\alpha\beta} + \mathcal{L}_X g_{\alpha\beta}) \\ & + 2V\mathrm{tr}_b \Big(\mathrm{Hess}_{\alpha\beta} N - N\mathrm{Ric}_{\alpha\beta} + 2N(K \circ K)_{\alpha\beta} - NHK_{\alpha\beta} \Big) \\ & + 2V\mathrm{tr}_b \Big(\mathcal{L}_X K_{\alpha\beta} \Big) \Big] \, dA_{\mathbb{S}^2_r}.\end{aligned}$$

Note that the asymptotics considered earlier are *not* preserved upon evolution, e.g.,

$$\mathcal{L}_X \dot{g}_{\alpha\beta} = {}^g\mathbf{m}_{\alpha\beta} + 2\frac{{}^g\tilde{\mathbf{m}}_{\alpha\beta}}{r} + O_1\left(r^{-2}\right),$$

and

$$\mathcal{L}_X \dot{g}_{rr} = \frac{\mathbf{m}_{rr}}{r^4} + 2\frac{\tilde{\mathbf{m}}_{rr}}{r^5} + O_1\left(r^{-6}\right)$$

Note that the asymptotics considered earlier are *not* preserved upon evolution, e.g.,

$$\mathcal{L}_X \dot{g}_{\alpha\beta} = {}^g\mathbf{m}_{\alpha\beta} + 2\frac{{}^g\tilde{\mathbf{m}}_{\alpha\beta}}{r} + O_1\left(r^{-2}\right),$$

and

$$\mathcal{L}_X \dot{g}_{rr} = \frac{\mathbf{m}_{rr}}{r^4} + 2\frac{\tilde{\mathbf{m}}_{rr}}{r^5} + O_1\left(r^{-6}\right)$$

After several cancellations, we obtain

$$\frac{\partial}{\partial\tau}E(\tau) = \frac{1}{16\pi}\lim_{r\to\infty}\int_{\mathbb{S}_r^2}\left(-\frac{4}{r^2}|{}^0\mathbf{g} + {}^0\mathbf{p}|_\sigma^2 + \frac{2}{r^2}\mathrm{div}_\sigma \mathbf{p}_{r\alpha} + \frac{1}{r^2}\mathrm{div}_\sigma \mathbf{g}_{r\alpha}\right) dA_{\mathbb{S}_r^2}$$
$$+ \frac{1}{16\pi}\lim_{r\to\infty}\int_{\mathbb{S}_r^2}\left(\frac{4}{r^2}\mathrm{tr}_\sigma {}^g\tilde{\mathbf{m}} + \frac{4}{r^2}\mathrm{tr}_\sigma\left({}^g\tilde{\mathbf{m}} + {}^k\tilde{\mathbf{m}}\right)\right) dA_{\mathbb{S}_r^2}.$$

Using our hypothesis, we obtain the result.

Linear Momentum: The statement and proof for the evolution of the linear momentum is entirely analogous, except for the additional condition that the following one forms are σ-divergences of symmetric tracefree two-tensors ${}^{g}\mathbf{T}_{\alpha\beta}$ and ${}^{k}\mathbf{T}_{\alpha\beta}$:

$$\mathbf{g}_{r\alpha} = \mathrm{div}_{\sigma}{}^{g}\mathbf{T}_{\alpha\beta}, \quad \mathbf{p}_{r\alpha} = \mathrm{div}_{\sigma}{}^{k}\mathbf{T}_{\alpha\beta},$$

Linear Momentum: The statement and proof for the evolution of the linear momentum is entirely analogous, except for the additional condition that the following one forms are σ-divergences of symmetric tracefree two-tensors ${}^g\mathbf{T}_{\alpha\beta}$ and ${}^k\mathbf{T}_{\alpha\beta}$:

$$\mathbf{g}_{r\alpha} = \mathrm{div}_\sigma {}^g\mathbf{T}_{\alpha\beta}, \quad \mathbf{p}_{r\alpha} = \mathrm{div}_\sigma {}^k\mathbf{T}_{\alpha\beta},$$

leading to

$$\frac{\partial}{\partial\tau} P^i(\tau) = -\frac{1}{4\pi} \int_{\mathbb{S}^2} \left| {}^0\mathbf{g} + {}^0\mathbf{p} \right|_\sigma^2 x^i \, dA_{\mathbb{S}^2}.$$

E-P Chargeability

The asymptotic structure of an asymptotically hyperboloidal initial data set is not preserved under the Einstein evolution equations.

E-P Chargeability

The asymptotic structure of an asymptotically hyperboloidal initial data set is not preserved under the Einstein evolution equations.

Consider the asymptotics of the $\tau = 1$ leaf of the foliation, that we denote by $(M, \bar{g}, \bar{K})$ obtained from evolution of the $\tau = 0$ AH IDS (M, g, K, ρ, J):

$$\bar{g}_{rr} = g_{rr} + \frac{\mathfrak{f}(u^A)}{r^4},$$

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + 2r\left({}^0\mathbf{g}_{\alpha\beta} + {}^0\mathbf{p}_{\alpha\beta}\right) + \mathfrak{a}_{\alpha\beta} + \frac{1}{r}\mathfrak{b}_{\alpha\beta},$$

$$\bar{K}_{\alpha\beta} = K_{\alpha\beta} - \mathfrak{f}(u^A)\sigma_{\alpha\beta} - \frac{2}{r}|{}^0\mathbf{g} + {}^0\mathbf{p}|^2_\sigma \sigma_{\alpha\beta},$$

where $\mathfrak{f} \in C^2(\mathbb{S}^2)$, while $\mathfrak{a}_{\alpha\beta} \in C^2(\mathbb{S}^2)$, and $\mathfrak{b}_{\alpha\beta} \in C^1(\mathbb{S}^2)$, with $\mathfrak{b}_{\alpha\beta}$ tracefree.

The energy of the evolved initial data $\left(M, \bar{g}, \bar{K}\right)$ is then given by

$$E(\tau = 1) = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2_r} \left(2\mathbf{m}_{rr} + 3\mathrm{tr}_\sigma{}^g\mathbf{m} + 2\mathrm{tr}_\sigma{}^k\mathbf{m} - 4|{}^0\mathbf{g} + {}^0\mathbf{p}|^2_\sigma\right) dA_{\mathbb{S}^2_r}.$$

Therefore, there are families of asymptotics other than the ones defined earlier that admit well-defined notions of energy and linear momentum.

E-P Chargeability

We say that (M, g, K, ρ, J) is *E-P chargeable* if there exist a spacelike hypersurface in Minkowski (M_0, g_0, K_0), compact sets $\mathcal{K}_0 \subset M_0, \mathcal{K} \subset M$, and a diffeomorphism at infinity

$$\Psi : M_0 \setminus \mathcal{K}_0 \to M \setminus \mathcal{K}$$

such that the following integrals converge:

$$\begin{aligned} m(\dot{g}, \dot{K}, \mathcal{V}) = \lim_{r\to\infty} \frac{1}{16\pi} \oint \Big[& V\big(\mathrm{div}_0 \dot{g} - d\mathrm{tr}_0 \dot{g}\big) - \iota_{\nabla V}\dot{g} + (\mathrm{tr}_0 \dot{g}) dV \\ & + 2\big(\iota_\alpha \dot{K} - (\mathrm{tr}_0 \dot{K})\alpha)\big) + (\mathrm{tr}_0 \dot{g})\iota_\alpha K_0 \\ & + \langle K_0, \dot{g}\rangle_0 \alpha - 2\iota_\alpha(\dot{g} \circ K_0)\Big](n) dA_{\mathbb{S}^2_r}, \end{aligned}$$

for any $\mathcal{V} \in \{\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$, where $\mathcal{V}_0, \mathcal{V}_i, i = 1, 2, 3$ are the KIDs on (M_0, g_0, K_0) corresponding to the time and spatial translations in Minkowski. Here, $dA_{\mathbb{S}^2_r}$ is the area element on the sphere of radius r in (M_0, g_0), and n is the outward unit normal to $\mathbb{S}^2_r$ within the bacgkround manifold. As before, $\dot{g} := \Psi^* g - g_0$ and $\dot{K} := \Psi^* K - K_0$.

- It is clear that $(M, \bar{g}, \bar{K})$ is E-P chargeable.

- It is clear that $(M, \bar{g}, \bar{K})$ is E-P chargeable.
- Evolving $(M, \bar{g}, \bar{K})$ leads to an initial data set that is E-P chargeable.

- It is clear that $(M, \bar{g}, \bar{K})$ is E-P chargeable.
- Evolving $(M, \bar{g}, \bar{K})$ leads to an initial data set that is E-P chargeable.
- E-P chargeability is preserved under evolution by the Einstein equations if natural conditions are imposed on the subleading terms of $\dot{g}$ and p.

Observations

- We observe that the fluxes for energy and linear momentum in our hyperboloidal approach closely mirror those in the Bondi–Sachs (null) formalism.

[10] e.g., Andersson, L., Chruściel, P. T.: On "hyperboloidal" Cauchy data for vacuum Einstein equations and obstructions to smoothness of Scri (1994)

Observations

- We observe that the fluxes for energy and linear momentum in our hyperboloidal approach closely mirror those in the Bondi–Sachs (null) formalism.
- The term ${}^{0}\mathbf{g}_{\alpha\beta} + {}^{0}\mathbf{p}_{\alpha\beta}$ plays a role akin to the News tensor in Bondi-Sachs coordinates.

[10] e.g., Andersson, L., Chruściel, P. T.: On "hyperboloidal" Cauchy data for vacuum Einstein equations and obstructions to smoothness of Scri (1994)

Observations

- We observe that the fluxes for energy and linear momentum in our hyperboloidal approach closely mirror those in the Bondi–Sachs (null) formalism.
- The term ${}^{0}\mathbf{g}_{\alpha\beta} + {}^{0}\mathbf{p}_{\alpha\beta}$ plays a role akin to the News tensor in Bondi-Sachs coordinates.
- The *shear tensor* is given by [10]:

$$\boldsymbol{\Xi}_{\alpha\beta} = \left(A_{\alpha\beta} - \frac{1}{2} h^{\rho\delta} A_{\rho\delta} h_{\alpha\beta} \right) \pm \left(\lambda_{\alpha\beta} - \frac{1}{2} h^{\rho\delta} \lambda_{\rho\delta} h_{\alpha\beta} \right).$$

[10] e.g., Andersson, L., Chruściel, P. T.: On "hyperboloidal" Cauchy data for vacuum Einstein equations and obstructions to smoothness of Scri (1994)

Observations

- We observe that the fluxes for energy and linear momentum in our hyperboloidal approach closely mirror those in the Bondi–Sachs (null) formalism.
- The term ${}^{0}\mathbf{g}_{\alpha\beta} + {}^{0}\mathbf{p}_{\alpha\beta}$ plays a role akin to the News tensor in Bondi-Sachs coordinates.
- The *shear tensor* is given by [10]:

$$\boldsymbol{\Xi}_{\alpha\beta} = \left(A_{\alpha\beta} - \frac{1}{2} h^{\rho\delta} A_{\rho\delta} h_{\alpha\beta} \right) \pm \left(\lambda_{\alpha\beta} - \frac{1}{2} h^{\rho\delta} \lambda_{\rho\delta} h_{\alpha\beta} \right).$$

- A generic asymptotically hyperboloidal IDS considered earlier will not be shear free.

[10] e.g., Andersson, L., Chruściel, P. T.: On “hyperboloidal” Cauchy data for vacuum Einstein equations and obstructions to smoothness of Scri (1994)

Outlook

1. Study the evolution of energy and linear momentum with respect to more general observers

Outlook

1. Study the evolution of energy and linear momentum with respect to more general observers
   - What are the conditions that would preserve E-P chargeability?

Outlook

1. Study the evolution of energy and linear momentum with respect to more general observers
    - What are the conditions that would preserve E-P chargeability?
2. Study the Hamiltonian charges and seek out discrepancies

Outlook

1. Study the evolution of energy and linear momentum with respect to more general observers
    - What are the conditions that would preserve E-P chargeability?
2. Study the Hamiltonian charges and seek out discrepancies
3. Other conserved quantities - angular momentum and center of mass

Outlook

1. Study the evolution of energy and linear momentum with respect to more general observers
    - What are the conditions that would preserve E-P chargeability?
2. Study the Hamiltonian charges and seek out discrepancies
3. Other conserved quantities - angular momentum and center of mass
4. Matter models

Thank you! :)

Questions and comments are welcome!