A spacetime interpretation to the confluent Heun functions in Black Hole Perturbation Theory

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Objectives

- Introduction to **Black Hole Perturbation Theory (BHPT)**;
- Introduce the **confluent Heun equation (CHE)** and discuss its solutions:
  - The general form of the CHE;
  - The non-symmetrical canonical form of the CHE.
- The Teukolsky master equation for the **Kerr family of spacetimes**;
- The **radial Teukolsky equation** as a confluent Heun equation:
  - Introduce the concept of **Heun slice**.
- The radial Teukolsky equation in compactified hyperboloidal coordinates within the Minimal Gauge:
  - Radial fixing gauge;
  - Cauchy fixing gauge.
- Conclusions.

Black Hole Perturbation Theory

The study of black hole perturbations has a long history, beginning with the work of Regge and Wheeler (1956) for Schwarzschild black holes, and later extended by Zerilli (1970), Vishveshwara (1970), Press (1971), Chandrasekhar and Detweiler (1975), and Teukolsky (1972, 1973, 1974) for Kerr black holes.

Why study black hole perturbations?

- Stability Analysis: Determining whether a black hole remains stable under small disturbances;
- Black Hole Ringdown: Describing how a perturbed black hole settles down by emitting gravitational waves;
- Gravitational Wave Astronomy: Understanding signals detected by LIGO and Virgo from black hole mergers;

and many more.

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One approach is to linearise the Einstein field equations around the Kerr metric and derive a set of perturbation equations.

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Consequence: This method leads to perturbation equations, but these do not fully decouple.

Solution: Teukolsky (1972-1974) overcame this issue by using the Newman-Penrose formalism and deriving perturbation equations for the **Weyl scalars** Ψ_0 and Ψ_4 rather than the full metric. This approach led to the **Teukolsky master equation**:

$$\mathcal{T}_{\mathfrak{s}}\Psi = 0.$$

- It decouples perturbations, making them easier to solve;
- It separates into radial and angular equations, which can be solved independently;
- The radial equation can be written as a **confluent Heun equation**.

The Confluent Heun Equation

In the past decade, there has been an increase of interest in the Heun functions in the context of BHPT, Hortacsu (2011), Vieira (2016), Noda (2022), London (2020)-(2023).

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The **Heun equation** is a second-order linear differential equation with four regular singular points (y_0, y_1, y_2, ∞). When two singularities are brought into coincidence $(y_1 = y_2)$ and ∞ is an irregular singular point the Heun equation becomes a **confluent Heun equation**.

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The **general form** of the confluent Heun equation is

$$\frac{d^2}{dy^2}Y(y) + \left(\sum_{i=0}^{1} \frac{A_i}{y - y_i} + E\right) \frac{d}{dy}Y(y) + \left(\sum_{i=0}^{1} \frac{C_i}{y - y_i} + \sum_{i=0}^{1} \frac{B_i}{(y - y_i)^2} + D\right) Y(y) = 0,$$

with $y \in [0, \infty)$. The regular singular points y_0 and y_1 are arbitrary, and so are the coefficients A_i, B_i, C_i, D, E (Ronveaux, 1995).

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The two linearly independent solutions are:

- **Locally** around y_i by
$$Y(y) \sim \mathcal{K}_{i+}(y-y_i)^{\varrho_i^+} + \mathcal{K}_{i-}(y-y_i)^{\varrho_i^-};$$
- **Asymptotically** by an expansion for $y \to \infty$ via
$$Y(y) \sim \mathcal{K}_{\infty+}\frac{e^{y\varsigma^+}}{y^{\eta^+}} + \mathcal{K}_{\infty-}\frac{e^{y\varsigma^-}}{y^{\eta^-}}.$$

The $\mathcal{K}_\pm$ are constants and the characteristic exponents are:

$$\varrho_i^\pm = \frac{1-A_i}{2} \pm \sqrt{\left(\frac{1-A_i}{2}\right)^2 - B_i}, \qquad \varsigma^\pm = -\frac{E}{2} \pm \frac{1}{2}\sqrt{E^2-4D},$$

$$\eta^\pm = \sum_{i=0}^{1}\left(\frac{A_i}{2} \pm \frac{C_i - EA_i/2}{\sqrt{E^2-4D}}\right).$$

The non-symmetrical canonical form of the CHE

The **non-symmetrical canonical form** of the CHE arises by mapping the regular singular points y_0 and y_1 into $z_0 = 0$ and $z_1 = 1$ via

$$z = \frac{y - y_0}{y_1 - y_0},$$

and via an **s-homotopic transformation**

$$Y(y(z)) = e^{\nu z} z^{\mu_0} (z-1)^{\mu_1} Z(z),$$

with the parameters

$$\mu_i = \frac{1 - A_i}{2} \pm \sqrt{\left(\frac{1 - A_i}{2}\right)^2 - B_i}, \quad \nu = \frac{y_1 - y_0}{2} \left(-E \pm \sqrt{E^2 - 4D}\right).$$

Then, the confluent Heun equation takes the form

$$\boxed{\frac{d^2 Z(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right) \frac{dZ(z)}{dz} + \frac{\alpha z - q}{z(z-1)} Z(z) = 0.}$$

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The function $Z(z)$'s behaviour around the regular singular points and asymptotically is equivalent to the solution of the general CHE but with characteristic exponents:

$$\varrho_0^+ = 1 - \gamma, \quad \varrho_1^+ = 1 - \delta, \quad \varrho_0^- = \varrho_1^- = 0,$$
$$\varsigma^+ = 0, \quad \varsigma^- = -\epsilon, \quad \eta^+ = \frac{\alpha}{\epsilon}, \quad \eta^- = \gamma + \delta - \frac{\alpha}{\epsilon}.$$

The Kerr family of spacetimes

The Kerr spacetime is a 2-parameter family of solutions to the vacuum Einstein field equations

$$\boldsymbol{Ric}[g_{ab}] = 0,$$

where g_{ab} is a 4-dimensional Lorentzian metric, describing a stationary, axisymmetric rotating black hole.

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In the standard **Boyer-Lindquist coordinates** $x^\mu = (t, r, \theta, \phi)$ the Kerr line element takes the form

$$\begin{aligned} \mathrm{d}s^2 = \quad & -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)\mathrm{d}t^2 - \frac{4Mar}{r^2 + a^2\cos^2\theta}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{r^2 + a^2\cos^2\theta}{\Delta(r)}\,\mathrm{d}r^2 \\ & + (r^2 + a^2\cos^2\theta)\,\mathrm{d}\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r}{r^2 + a^2\cos^2\theta}\sin^2\theta\right)\mathrm{d}\phi^2, \end{aligned}$$

where the parameters M and a relate to the black hole's mass and angular momentum.

The condition $\Delta(r) = r^2 - 2Mr + a^2 = 0$ determines the location of the event r_h and Cauchy r_c horizons

$$r_\mathrm{h} = M\left(1 + \sqrt{1 - \frac{a^2}{M^2}}\right), \qquad r_\mathrm{c} = M\left(1 - \sqrt{1 - \frac{a^2}{M^2}}\right).$$

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The perturbation dynamics are dictated by the Teukolsky master equation.

The **Teukolsky master equation** for ${}_{\mathfrak{s}}\Psi(x^a)$ is:

$$
\begin{aligned}
&-\left[\frac{(r^2+a^2)^2}{\Delta(r)} - a^2\sin^2\theta\right]{}_{\mathfrak{s}}\Psi_{,tt}(x^a) - \frac{4Mar}{\Delta(r)}\,{}_{\mathfrak{s}}\Psi_{,t\phi}(x^a) - \left[\frac{a^2}{\Delta(r)} - \frac{1}{\sin^2\theta}\right]{}_{\mathfrak{s}}\Psi_{,\phi\phi}(x^a)\\
&+\Delta^{-\mathfrak{s}}(r)\,\partial_r\left(\Delta^{\mathfrak{s}+1}(r)\,{}_{\mathfrak{s}}\Psi_{,r}(x^a)\right) + 2\mathfrak{s}\left[\frac{M(r^2-a^2)}{\Delta(r)} + (r+\mathrm{i}a\cos\theta)\right]{}_{\mathfrak{s}}\Psi_{,t}(x^a)\\
&+2\mathfrak{s}\left[\frac{a(r-M)}{\Delta(r)} + \mathrm{i}\frac{\cos\theta}{\sin^2\theta}\right]{}_{\mathfrak{s}}\Psi_{,\phi}(x^a) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,{}_{\mathfrak{s}}\Psi_{,\theta}(x^a)\right) - \mathfrak{s}(\mathfrak{s}\cot^2\theta - 1){}_{\mathfrak{s}}\Psi(x^a) = 0.
\end{aligned}
$$

where $\mathfrak{s}$ is the spin describing the type of perturbation.

The Teukolsky Master Equation

The **Teukolsky master equation** for ${}_{\mathfrak{s}}\Psi(x^a)$ is:

$$-\left[\frac{(r^2+a^2)^2}{\Delta(r)} - a^2\sin^2\theta\right] {}_{\mathfrak{s}}\Psi_{,tt}(x^a) - \frac{4Mar}{\Delta(r)}\,{}_{\mathfrak{s}}\Psi_{,t\phi}(x^a) - \left[\frac{a^2}{\Delta(r)} - \frac{1}{\sin^2\theta}\right] {}_{\mathfrak{s}}\Psi_{,\phi\phi}(x^a)$$
$$+\Delta^{-\mathfrak{s}}(r)\,\partial_r\left(\Delta^{\mathfrak{s}+1}(r)\,{}_{\mathfrak{s}}\Psi_{,r}(x^a)\right) + 2\mathfrak{s}\left[\frac{M(r^2-a^2)}{\Delta(r)} + (r + \mathrm{i}a\cos\theta)\right] {}_{\mathfrak{s}}\Psi_{,t}(x^a)$$
$$+2\mathfrak{s}\left[\frac{a(r-M)}{\Delta(r)} + \mathrm{i}\frac{\cos\theta}{\sin^2\theta}\right] {}_{\mathfrak{s}}\Psi_{,\phi}(x^a) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,{}_{\mathfrak{s}}\Psi_{,\theta}(x^a)\right) - \mathfrak{s}(\mathfrak{s}\cot^2\theta - 1)\,{}_{\mathfrak{s}}\Psi(x^a) = 0.$$

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One can separate the Teukolsky equation in the **frequency domain** by introducing a **Fourier decomposition**

$${}_{\mathfrak{s}}\Psi(t,r,\theta,\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{\ell=|\mathfrak{s}|}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\mathrm{i}\omega t}e^{\mathrm{i}m\phi}\,{}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r)\,{}_{\mathfrak{s}}S_{\ell m}(\omega;\theta)\mathrm{d}\omega.$$

This leads to two second-order ordinary differential equations, the angular Teukolsky equation and the radial Teukolsky equation.

We will focus on the **radial Teukolsky equation**:

$$\Delta^{-\mathfrak{s}}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{\mathfrak{s}+1}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\right){}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r) + \left(2i\mathfrak{s}\omega r - a^2\omega^2 - A_{\ell m}\right){}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r)$$
$$+\frac{(\omega\Sigma_0)^2 - 4Mam\omega r + a^2m^2 + 2i\mathfrak{s}\left[am(r-M) - M\omega(r^2-a^2)\right]}{\Delta(r)}\,{}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r) = 0.$$

The Radial Teukolsky Equation as a Confluent Heun Equation

The **radial Teukolsky equation** can be written in the form of a **general CHE**:

$$\frac{d^2}{dr^2}\,{}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r) + \left(\sum_{i=0}^{1}\frac{A_i}{r-r_i} + E\right)\frac{d}{dr}\,{}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r)$$
$$+\left(\sum_{i=0}^{1}\frac{C_i}{r-r_i} + \sum_{i=0}^{1}\frac{B_i}{(r-r_i)^2} + D\right){}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r) = 0,$$

with regular singular points $r_0 = r_{\rm c}$ and $r_1 = r_{\rm h}$, while the irregular singular point is $r = \infty$.

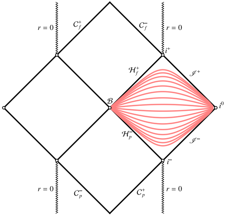
Two linearly independent solutions in the asymptotic region are provided by:

- A **local** solution around r_h:
$$ {}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r) \sim \mathcal{K}_{\mathcal{H}_{\rm p}}\,(r-r_{\rm h})^{\varrho_{\mathcal{H}_{\rm p}}} \quad + \quad \mathcal{K}_{\mathcal{H}_{\rm f}}\,(r-r_{\rm h})^{\varrho_{\mathcal{H}_{\rm f}}};$$
- An **asymptotic** solution via an expansion for $r \to \infty$
$$ {}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r) \sim \mathcal{K}_{\infty^+}\, r^{-\eta^+} e^{\varsigma^+ r} \quad + \quad \mathcal{K}_{\infty^-}\, r^{-\eta^-} e^{\varsigma^- r}.$$

where $\mathcal{K}_\pm$ can be chosen as for the “in”, “out”, “up” and “down” solutions.

The characteristic exponents are:

$$\varrho_{\mathcal{C}_{\rm f}} = -\frac{{\rm i}\omega}{2\varkappa_c} + \frac{{\rm i}m\Omega_c}{2\varkappa_c} - \mathfrak{s}, \qquad \varrho_{\mathcal{C}_{\rm p}} = \frac{{\rm i}\omega}{2\varkappa_c} - \frac{{\rm i}m\Omega_c}{2\varkappa_c},$$
$$\varrho_{\mathcal{H}_{\rm p}} = \frac{{\rm i}\omega}{2\varkappa_h} - \frac{{\rm i}m\Omega_h}{2\varkappa_h}, \qquad \varrho_{\mathcal{H}_{\rm f}} = -\frac{{\rm i}\omega}{2\varkappa_h} + \frac{{\rm i}m\Omega_h}{2\varkappa_h} - \mathfrak{s},$$
$$\varsigma^\pm = \pm{\rm i}\omega, \quad \eta^+ = 1 - 2{\rm i}M\omega + 2\mathfrak{s}, \qquad \eta^- = 1 + 2{\rm i}M\omega.$$



Studies of the radial Teukolsky equation as a confluent Heun equation usually present it in the **non-symmetrical canonical CHE form**:

$$\frac{d^2}{dz^2}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right)\frac{d}{dz}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z)+\frac{\alpha z-q}{z(z-1)}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z)=0.$$

This form follows from the **coordinate change**

$$z=\frac{r-r_{\mathrm{c}}}{r_{\mathrm{h}}-r_{\mathrm{c}}},$$

mapping $r=\{r_{\mathrm{c}},r_{\mathrm{h}},\infty\}$ into $z=\{0,1,\infty\}$, together with an **s-homotopic transformation**

$${}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega;r(z))=z^{\mu_c}(z-1)^{\mu_h}e^{\nu z}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z).$$

The characteristic exponents are:

$$\mu_c^{+}=-\mathfrak{s}-\frac{\mathrm{i}\,\omega-\mathrm{i}m\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}},\quad \mu_c^{-}=\frac{\mathrm{i}\,\omega-\mathrm{i}m\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}},$$

$$\mu_h^{+}=-\mathfrak{s}-\frac{\mathrm{i}\,\omega-\mathrm{i}m\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}},\quad \mu_h^{-}=\frac{\mathrm{i}\,\omega-\mathrm{i}m\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}},$$

$$\nu^{+}=\mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2),\quad \nu^{-}=-\mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2).$$

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$$\frac{d^2}{dz^2}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{d}{dz}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) + \frac{\alpha z - q}{z(z-1)}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) = 0.$$

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The characteristic exponents are:

$$\mu_c^+ = -\mathfrak{s} - \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}}, \quad \mu_c^- = \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}},$$
$$\mu_h^+ = -\mathfrak{s} - \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}}, \quad \mu_h^- = \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}},$$
$$\nu^+ = \mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2), \quad \nu^- = -\mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2).$$

There is a total of 8 combinations of $\{\mu_c, \mu_h, \nu\}$!

The Radial Teukolsky Equation in the canonical CHE form

Studies of the radial Teukolsky equation as a confluent Heun equation usually present it in the **non-symmetrical canonical CHE form**:

$$\frac{d^2}{dz^2}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right)\frac{d}{dz}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z)+\frac{\alpha z-q}{z(z-1)}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z)=0.$$

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$$\nu^+=\mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2),\quad \nu^-=-\mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2).$$

There is a total of 8 combinations of $\{\mu_c,\mu_h,\nu\}$!

- **What is the effect of the s-homotopic transformation?**

The Radial Teukolsky Equation in the canonical CHE form

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$$\frac{d^2}{dz^2}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{d}{dz}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) + \frac{\alpha z - q}{z(z-1)}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) = 0.$$

This form follows from the **coordinate change**

$$z = \frac{r - r_{\mathrm{c}}}{r_{\mathrm{h}} - r_{\mathrm{c}}},$$

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The characteristic exponents are:

$$\mu_c^+ = -\mathfrak{s} - \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}}, \quad \mu_c^- = \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}},$$
$$\mu_h^+ = -\mathfrak{s} - \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}}, \quad \mu_h^- = \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}},$$
$$\nu^+ = \mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2), \quad \nu^- = -\mathrm{i}\,r_{\mathrm{h}}\omega\,(1-\kappa^2).$$

There is a total of 8 combinations of $\{\mu_c, \mu_h, \nu\}$**!**

- **What is the effect of the s-homotopic transformation?**

It modifies the behaviour of ${}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r)$ around the singular points by

$$\tilde{\varrho}_{\mathcal{C}_{\mathrm{p/f}}} = \varrho_{\mathcal{C}_{\mathrm{p/f}}} - \mu_c, \quad \tilde{\varrho}_{\mathcal{H}_{\mathrm{p/f}}} = \varrho_{\mathcal{H}_{\mathrm{p/f}}} - \mu_h, \quad \tilde{\eta}_{\infty^\pm} = \eta_{\infty^\pm} + \mu_c + \mu_h, \quad \tilde{\varsigma}_{\infty^\pm} = \varsigma_{\infty^\pm} - \frac{\nu}{(r_h - r_c)}.$$

Studies of the radial Teukolsky equation as a confluent Heun equation usually present it in the **non-symmetrical canonical CHE form**:

$$\frac{d^2}{dz^2}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{d}{dz}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) + \frac{\alpha z - q}{z(z-1)}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z) = 0.$$

This form follows from the **coordinate change**

$$z = \frac{r - r_{\rm c}}{r_{\rm h} - r_{\rm c}},$$

mapping $r = \{r_{\rm c}, r_{\rm h}, \infty\}$ into $z = \{0, 1, \infty\}$, together with an **s-homotopic transformation**

$${}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r(z)) = z^{\mu_c}(z-1)^{\mu_h} e^{\nu z}\,{}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega;z).$$

The characteristic exponents are:

$$\mu_c^+ = -\mathfrak{s} - \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\rm c}}{2\varkappa_{\rm c}}, \quad \mu_c^- = \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\rm c}}{2\varkappa_{\rm c}},$$

$$\mu_h^+ = -\mathfrak{s} - \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\rm h}}{2\varkappa_{\rm h}}, \quad \mu_h^- = \frac{\mathrm{i}\,\omega - \mathrm{i}m\Omega_{\rm h}}{2\varkappa_{\rm h}},$$

$$\nu^+ = \mathrm{i}\,r_{\rm h}\omega\,(1-\kappa^2), \quad \nu^- = -\mathrm{i}\,r_{\rm h}\omega\,(1-\kappa^2).$$

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- **What is the spacetime interpretation of the 8 configurations** $\{\mu_c, \mu_h, \nu\}$**?**

Heun slices

We introduce a generic coordinate system $\tilde{x}^\mu = (\tilde{t}, r, \theta, \tilde{\phi})$ related to the BL coordinates via

$$t = \tilde{t} - h(r),$$
$$\phi = \tilde{\phi} - \xi(r)$$

and a transformation for the Teukolsky master function ${}_{\mathfrak{s}}\Psi(x^\mu)$ of the form

$${}_{\mathfrak{s}}\tilde{\Psi}(\tilde{x}^\mu) = \zeta^{\mathfrak{s}}(r)\,{}_{\mathfrak{s}}\Psi(x^\mu).$$

The Fourier decomposition of ${}_{\mathfrak{s}}\tilde{\Psi}(\tilde{t}, r, \theta, \tilde{\phi})$ into the frequency domain shows that

$$\underbrace{{}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r)}_{\text{BL radial function}} = \zeta(r)^{-\mathfrak{s}}\, e^{-\mathrm{i}\omega h(r)} e^{\mathrm{i}m\xi(r)} \underbrace{{}_{\mathfrak{s}}\tilde{\mathcal{R}}_{\ell m}(\omega; r)}_{\text{New radial function}}$$

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A direct comparison against the s-homotopic transformation

$${}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r(z)) = z^{\mu_c}(z-1)^{\mu_h} e^{\nu z}\, {}_{\mathfrak{s}}\mathrm{R}_{\ell m}(\omega; z) \quad \text{where} \quad z = \frac{r - r_{\mathrm{c}}}{r_{\mathrm{h}} - r_{\mathrm{c}}}.$$

allows us to identify the functions $h(r)$, $\xi(r)$ and $\zeta(r)$ as modifications of $r^*(r)$, $\chi(r)$ and $\Delta(r)$ in the form

$$h(r) = a_\infty\, r + 2M\, b_\infty \ln\left(\frac{r}{r_{\mathrm{h}}}\right) + \frac{b_h}{2\varkappa_{\mathrm{h}}} \ln\left(1 - \frac{r_{\mathrm{h}}}{r}\right) + \frac{b_c}{2\varkappa_{\mathrm{c}}} \ln\left(1 - \frac{r_{\mathrm{c}}}{r}\right),$$
$$\xi(r) = c_\infty \ln\left(\frac{r}{r_{\mathrm{h}}}\right) + c_h \frac{\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}} \ln\left(1 - \frac{r_{\mathrm{h}}}{r}\right) + c_c \frac{\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}} \ln\left(1 - \frac{r_{\mathrm{c}}}{r}\right),$$
$$\zeta(r) = (r - r_{\mathrm{h}})^{d_h} (r - r_{\mathrm{c}})^{d_c}.$$

Heun slices

THE CENTER OF GRAVITY

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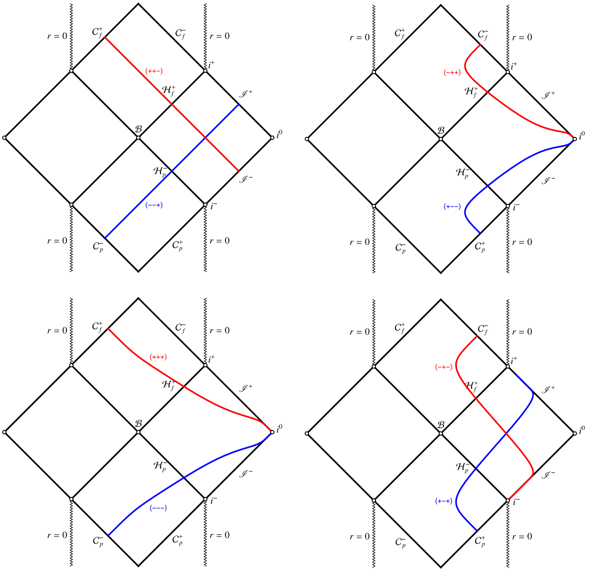
$$\xi(r) = c_\infty \ln\left(\frac{r}{r_{\mathrm{h}}}\right) + c_h \frac{\Omega_{\mathrm{h}}}{2\varkappa_{\mathrm{h}}} \ln\left(1 - \frac{r_{\mathrm{h}}}{r}\right) + c_c \frac{\Omega_{\mathrm{c}}}{2\varkappa_{\mathrm{c}}} \ln\left(1 - \frac{r_{\mathrm{c}}}{r}\right),$$
$$\zeta(r) = (r - r_{\mathrm{h}})^{d_h} (r - r_{\mathrm{c}})^{d_c}.$$

Each configuration is a choice of $\{\mu_c, \mu_h, \nu\}$.

Heun slices in the Carter-Penrose diagram of Kerr spacetimes



The $\pm$ correspond to a choice of $\{\mu_c, \mu_h, \nu\}$

$h(r)$, $\xi(r)$ and $\zeta(r)$ for

$$t = \tilde{t} - h(r),$$
$$\phi = \tilde{\phi} - \xi(r)$$

and

$${}_{\mathfrak{s}}\tilde{\Psi}(\tilde{x}^{\mu}) = \zeta^{\mathfrak{s}}(r)\,{}_{\mathfrak{s}}\Psi(x^{\mu}).$$

THE CENTER OF GRAVITY

The confluent Heun functions arise from a coordinate transformation related to BL coordinates via

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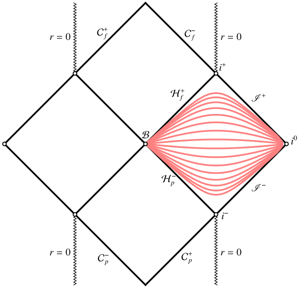
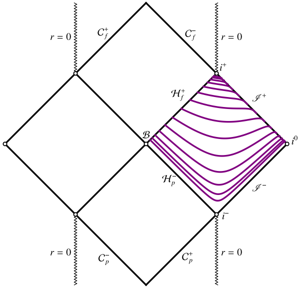
Does the radial Teukolsky equation, expressed in compactified hyperboloidal coordinates, take the form of a confluent Heun equation?



Boyer-Lindquist slices



Hyperboloidal slices

The Hyperboloidal Formalism

The hyperboloidal formalism consists of:

- A **coordinate transformation** $\bar{x}^a = (\bar{t}, \sigma, \theta, \bar{\phi})$ given by:

$$t = \lambda\big(\bar{t} - H(\sigma, \theta)\big),$$
$$\phi = \bar{\phi} - \bar{\chi}(\sigma),$$

  with $H(\sigma, \theta)$ a **height function**, along with a compactification of the radial coordinate:

$$r = \lambda \frac{\rho(\sigma)}{\sigma},$$

  mapping $r = \{0, r_{\rm c}, r_{\rm h}\}$ into $\sigma = \{\sigma_0, \sigma_{\rm c}, \sigma_{\rm h}\}$.

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- A **conformal re-scaling** of the Kerr metric:

$$\mathrm{d}\bar{s}^2 = \Omega(\sigma)^2\, \mathrm{d}s^2, \quad \Omega(\sigma) = \sigma/\lambda,$$

  with the conformal metric $\mathrm{d}\bar{s}^2$ regular in the domain $\sigma \in [0, \sigma_0)$.

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In these coordinates ${}_{\mathfrak{s}}\bar{\Psi}(\bar{x}^a)$, the original Teukolsky master function, is transformed as

$$\underbrace{{}_{\mathfrak{s}}\bar{\Psi}(\bar{x}^a)}_{\text{Boyer-Lindquist master function}} = \Omega\Delta^{-\mathfrak{s}} \underbrace{{}_{\mathfrak{s}}\Psi(x^a)}_{\text{Hyperboloidal master function}}$$

A Fourier decomposition of ${}_{\mathfrak{s}}\bar{\Psi}$ into the frequency domain shows that

$$\underbrace{{}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r(\sigma))}_{\text{Boyer-Lindquist radial function}} = \Omega\Delta^{-\mathfrak{s}}\, e^{-\mathrm{i}\omega\lambda H(\sigma)} e^{\mathrm{i}m\bar{\chi}} \underbrace{{}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega; \sigma)}_{\text{Conformal radial function}}$$

The Minimal Gauge

One considers the radial degree of freedom

$$\rho(\sigma) = \rho_0 + \rho_1 \sigma, \quad \rho_0 = \frac{r_{\rm h}}{\lambda} - \rho_1,$$

where the choice for ρ_0 ensures that the radial transformation maps $r_{\rm h}$ to $\sigma_{\rm h} = 1$.

The Cauchy horizon and singularity assume the values

$$\sigma_{\rm c} = \frac{1 - \lambda\rho_1/r_{\rm h}}{\kappa^2 - \lambda\rho_1/r_{\rm h}}, \quad \sigma_0 = 1 - \frac{r_{\rm h}}{\lambda\rho_1}.$$

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where the choice for ρ_0 ensures that the radial transformation maps r_h to $\sigma_\mathrm{h} = 1$.

The Cauchy horizon and singularity assume the values

$$\sigma_\mathrm{c} = \frac{1 - \lambda\rho_1/r_\mathrm{h}}{\kappa^2 - \lambda\rho_1/r_\mathrm{h}}, \quad \sigma_0 = 1 - \frac{r_\mathrm{h}}{\lambda\rho_1}.$$

One also introduces the dimensionless tortoise coordinates

$$\begin{aligned} x(\sigma) &= \frac{r^*(r(\sigma))}{\lambda} \\ &= x_\infty(\sigma) + x_\mathrm{h}(\sigma) + x_\mathrm{c}(\sigma) + x_\mathrm{cont}, \end{aligned}$$

so that

$$H(\sigma) = -x_\infty(\sigma) + x_\mathrm{h}(\sigma) + x_\mathrm{c}(\sigma).$$

The angular tortoise coordinate $\bar{\chi}(\sigma) = \chi(r(\sigma))$ is

$$\bar{\chi}(\sigma) = \frac{\Omega_h}{2\varkappa_h} \ln(1-\sigma) + \frac{\Omega_c}{2\varkappa_c} \ln\left(1 - \frac{\sigma}{\sigma_c}\right).$$

Eventually, **there remains only the parameter ρ_1 to fully fix the hyperboloidal foliation.**

The **radial fixing gauge** is characterised by setting $\rho_1 = 0$ so that

$$r(\sigma) = \frac{r_h}{\sigma}.$$

The radial compactification maps $r = \{0, r_c, r_h, \infty\}$ into $\sigma = \{\sigma_0, \sigma_c, \sigma_h, \sigma_\infty\}$ where $\sigma_0 \to \infty$.

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The radial compactification maps $r = \{0, r_c, r_h, \infty\}$ into $\sigma = \{\sigma_0, \sigma_c, \sigma_h, \sigma_\infty\}$ where $\sigma_0 \to \infty$.

The radial Teukolsky equation for ${}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega;\sigma)$ **is not** a confluent Heun equation because:

- The equation contains **three regular singular points** $\sigma = \{\sigma_0, \sigma_c, \sigma_h\}$ and **one irregular singular point** $\sigma_\infty = 0$;
- The resemblance of a confluent Heun equation when passing to σ is due to the absence of the coefficients associated to σ_0. However, this point is not an ordinary point of the equation but a regular singular point of the equation;
- The radial equation along a hyperboloidal slice expressed in terms of z is

$$z(\sigma) = \frac{1}{\sigma}\frac{1-\kappa^2\sigma}{1-\kappa^2} \longleftrightarrow \sigma(z) = \frac{1}{z+\kappa^2(1-z)}.$$

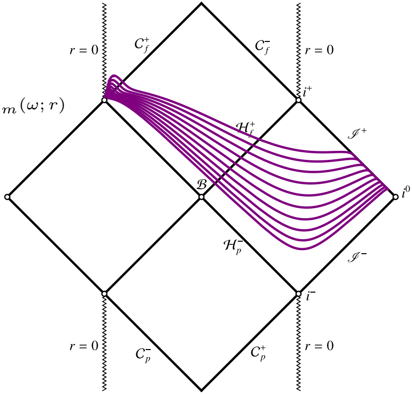
  so that the resulting equation *does not* assume the canonical form of the confluent Heun equation.

By looking at the **s-homotopic** transformation of the hyperboloidal radial function:

$$ {}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r) = \underline{r^{\bar{\mu}_\infty}}\, (r - r_{\rm h})^{\bar{\mu}_h}\, (r - r_{\rm c})^{\bar{\mu}_c}\, e^{\bar{\nu} r}\, {}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega; r) $$

the triad $(\bar{\mu}_c, \bar{\mu}_h, \bar{\nu})$ directly relates to the Heun configuration $(+, +, +)$. However, $\bar{\mu}_\infty \neq 0$ is absent in the case of any Heun configuration.

The coordinate transformation introduces a regular singular point corresponding to $r = 0$. Hence, **the radial Teukolsky equation written in terms of compactified hyperboloidal coordinate is not a confluent Heun equation**.



The Cauchy fixing

The **Cauchy fixing gauge** is characterised by setting $\rho_1 = \dfrac{r_\mathrm{c}}{\lambda}$ from which it follows

$$r(\sigma) = \frac{r_\mathrm{h} - r_\mathrm{c}}{\sigma} + r_\mathrm{c} \iff \sigma(r) = \frac{r_\mathrm{h} - r_\mathrm{c}}{r - r_\mathrm{c}},$$

mapping $r = r_\mathrm{c}$ to $\sigma = \sigma_\mathrm{c} \to \infty$. The surface $r = 0$ is mapped to a point $\sigma = \sigma_0$ which lies outside the domain $\sigma \in [0, \infty)$.

The radial Teukolsky equation for ${}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega;\sigma)$ is a confluent Heun equation in the form

$$\begin{aligned} \frac{d^2}{d\sigma^2}\,{}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega;\sigma) \quad &+ \quad \left(\sum_{i=0}^{1} \frac{A_i{}^{(\sigma)}}{\sigma - \sigma_i} + \sum_{j=1}^{2} \frac{E_j{}^{(\sigma)}}{(\sigma - \sigma_2)^j}\right) \frac{d}{d\sigma}\,{}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega;\sigma) \\ &+ \quad \left(\sum_{i=0}^{1}\left(\frac{C_i{}^{(\sigma)}}{\sigma - \sigma_i} + \frac{B_i{}^{(\sigma)}}{(\sigma - \sigma_i)^2}\right) + \sum_{j=1}^{4} \frac{D_j{}^{(\sigma)}}{(\sigma - \sigma_2)^j}\right) {}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega;\sigma) = 0 \end{aligned}$$

because:

- The differential equation has **two regular singular points** $\sigma = \{\sigma_h, \sigma_c\}$ and **one irregular singular point** $\sigma_\infty = 0$;
- The coefficients *do not* satisfy the CHE constraints ensuring that the point $\sigma_c \to \infty$ is not an ordinary point but a singular point of the equation;
- The radial Teukolsky equation one obtains by introducing z as

  $$z(\sigma) = \frac{1}{\sigma} \quad \longleftrightarrow \quad \sigma(z) = \frac{1}{z}.$$

  assumes the form of a canonical CHE with **two regular singular points** $z = \{z_c, z_h\}$ and an **one irregular singular point** $z = z_\infty$.

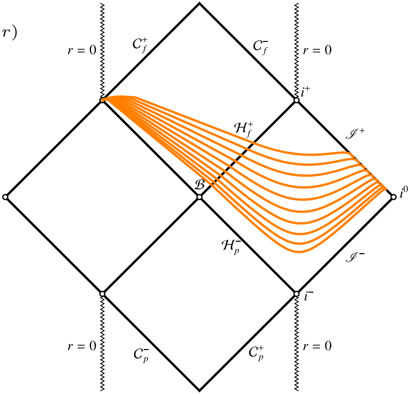
Comparison between the hyperboloidal slices in the Cauchy fixing with Heun slices

By looking at the **s-homotopic** transformation of the hyperboloidal radial Teukolsky equation:

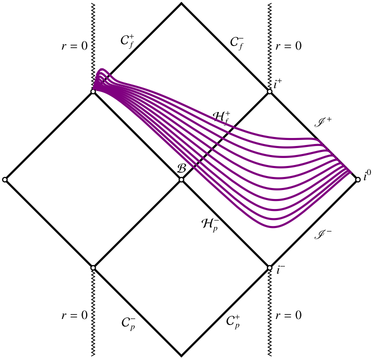
$$ {}_{\mathfrak{s}}\mathcal{R}_{\ell m}(\omega; r) = (r - r_{\rm h})^{\bar{\mu}_h} (r - r_{\rm c})^{\bar{\mu}_c} e^{\bar{\nu} r} {}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega; r) $$

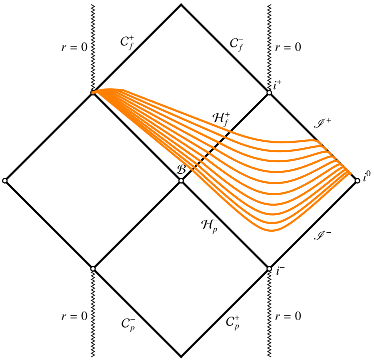
the triad $(\bar{\mu}_c, \bar{\mu}_h, \bar{\nu})$ directly relates to the Heun configuration $(+, +, +)$. However, in contrast to the radial fixing case $\bar{\mu}_\infty = 0$. So we have the same type of s-homotopic transformation one has for a CHE. Hence, **the radial Teukolsky equation written in terms of compactified hyperboloidal coordinates is a CHE**.

The s-homotopic transformation with coefficients $\{\bar{\mu}_c, \bar{\mu}_h, \bar{\nu}\}$ reproduces the Ansatz introduced by the Leaver (1985), which allows the spacetime interpretation of his approach as the frequency domain representation of Teukolsky equation in the hyperboloidal Cauchy fixing gauge.



Carter-Penrose diagrams representing hyperboloidal slices for the two gauge choices





If we compare the height functions:

$$H_{rf}(r) = -r + 2M\ln\left(\frac{r}{r_{\rm h}}\right) + \frac{1}{2\varkappa_h}\ln\left(1-\frac{r_{\rm h}}{r}\right) + \frac{1}{2\varkappa_c}\ln\left(1-\frac{r_{\rm c}}{r}\right),$$

$$H_{cf}(r) = -r - 2M\ln\left(\frac{r}{r_{\rm h}}\right) + \frac{1}{2\varkappa_h}\ln\left(1-\frac{r_{\rm h}}{r}\right) - \left(2M + \frac{1}{2\varkappa_h}\right)\ln\left(1-\frac{r_{\rm c}}{r}\right) + H_{\rm const}.$$

Hence, the choice of height function determines the presence of $\bar{\mu}_\infty \neq 0$ in the s-homotopic transformation with the introduction of an additional regular singular point $r = 0$ $(\sigma = \sigma_0)$ in the equation for ${}_{\mathfrak{s}}\bar{\mathcal{R}}_{\ell m}(\omega;\sigma)$.

Conclusions and future perspectives

1. Reviewed the use of Heun functions in solving the radial Teukolsky equation, showing that a spacetime-level coordinate transformation reproduces the effect of z and s-homotopic transformations.
2. Introduced the concept of **Heun slices**, defined via new spacetime coordinate systems, and connected their geometry to properties of Heun functions.
3. Demonstrated that the canonical confluent Heun equation form naturally arises from specific spacetime foliations, with configurations labelled by triplets of signs $(- - +)$, $(+ + -)$.
4. Highlighted the importance of the slicing configuration for the asymptotic behaviour of radial perturbations, linking characteristic exponents to properties of null infinity $(\mathscr{I}^+/\mathscr{I}^-)$, spatial infinity (i^0), and time infinity $(i^\pm)$.
5. Explored hyperboloidal coordinates, noting that some height function choices, like Cauchy fixing, compatible with the Minimal Gauge can lead to an radial Teukolsky equation in the form of a confluent Heun equation.

Thank you!

Any questions?