Nonlinear evolution of unstable charged de Sitter black holes with hyperboloidal formalism

Yu Tian

School of Physical Sciences, University of Chinese Academy of Sciences

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Outline

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Traditional $3+1$ foliation and hyperboloidal foliation

- For asymptotically flat space-times, traditional $3+1$ foliation does not cover the future null infinity $\mathscr{I}^+$, but some key physical quantities, *e.g.* the radiated energy and angular momentum of the gravitational wave or decay rates of the late-time tail of BH perturbations, should be calculated at $\mathscr{I}^+$.
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- In order to obtain such quantities, one needs to put an artificial boundary (and the corresponding BC) at a large distance and extract those quantities there (approximately).
- Hyperboloidal foliation (as a special kind of $3+1$) solves the above problems by covering $\mathscr{I}^+$ with a properly designed time slicing.

  [A. Zenginoğlu, arXiv:0712.4333]

Example: hyperboloidal coordinates for the Schwarzschild space-time

- The original Schwarzschild coordinates

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

$$f(r) = 1 - \frac{2M}{r}$$

- The in-going Eddington-Finkelstein coordinates (3 + 1, in a Kerr-Schild form)

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 + (1-f)(dt-dr)^2$$

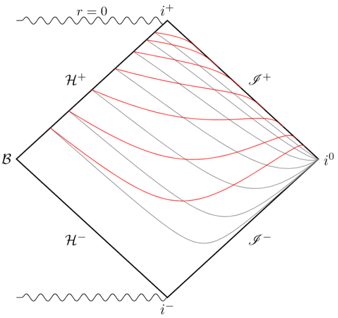


Figure: Comparison of foliations between the ingoing Eddington-Finkelstein and hyperboloidal coordinates of the Schwarzschild space-time

Example: hyperboloidal coordinates for the Schwarzschild space-time

- Taking the transformation through the height function technique

$$T = t - h(r), \qquad Z = \frac{L^2}{r}, \tag{1.1}$$

one can obtain the hyperboloidal coordinates:

$$\mathrm{d}s^2 = \frac{L^2}{Z^2}[-\mathrm{d}T^2 + (\mathrm{d}Z + H\mathrm{d}T)^2] + \frac{L^4}{Z^2}\mathrm{d}\Omega^2 \tag{1.2}$$

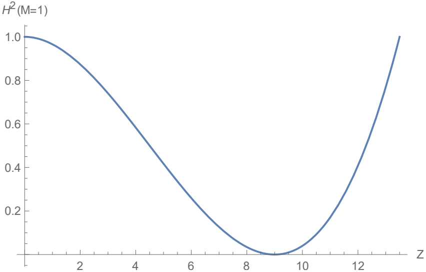
with L an appropriate length scale and the "shift" function

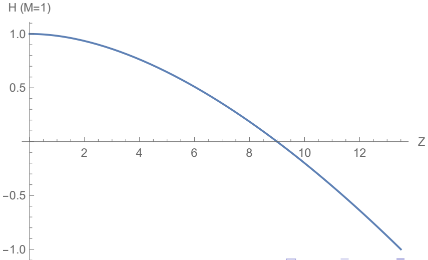
$$H(Z)^2 = 1 - \frac{Z^2}{L^2}f(r) = 1 - \frac{Z^2}{L^2} + \frac{2MZ^3}{L^4}.$$

At the boundary positions $H^2 = 1$, since either $Z = 0$ or $f(r) = 0$.

Example: hyperboloidal coordinates for the Schwarzschild space-time

- L is so chosen that
$$H(Z) = (\pm)\sqrt{1 - \frac{Z^2}{L^2} + \frac{2MZ^3}{L^4}}$$
can be a continuous real function decreasing monotonically from 1 to -1 in the range $Z \in [0, Z_h = \frac{L^2}{2M}]$, which means that H^2 should have a zero at $Z = Z_0$ and be positive elsewhere ($L = \sqrt{27}M$).





Problem: instability of the RN-dS black hole

- Hyperboloidal slices for black hole dynamics in the asymptotically flat case

  [O. Rinne, arXiv:0910.0139; A. Vañó-Viñuales, S. Husa & D. Hilditch, arXiv:1412.3827; ...]

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- How about black hole dynamics in the asymptotically dS case with the hyperboloidal formalism, where traditional methods does not even work?
- An interesting problem: In certain parameter range the RN-dS black hole under charged scalar perturbations is linearly unstable, but how will it evolve and what is the final state of this instability?

  [Z. Zhu et al., PRD (2014); Konoplya & Zhidenko, PRD (2014)]

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- A no-charged-scalar-hair theorem for static dS black holes has been established, which precludes the possibility of the formation of a hairy black hole.

  [Y.-P. An & L. Li, EPJC (2023)]

Outline

1. Introduction
2. Static solutions and quasi-normal modes of de Sitter (dS) black holes in hyperboloidal coordinates
3. Nonlinear evolution of unstable charged dS black holes in hyperboloidal coordinates
4. Summary and outlook

Hyperboloidal coordinates for the RN-dS space-time

- One can obtain a closed-form RN-dS solution in hyperboloidal coordinates (1.2), with

$$H(Z)^2 = 1 - \frac{Z^2}{L^2} + \frac{2M_0 Z^3}{L^4} - \frac{Q_0^2 Z^4}{L^6} + \frac{\Lambda L^2}{3},$$

  where M_0 and Q_0 are the total mass and charge of the black hole, respectively.

  [Z.-T. He, Q. Chen, YT, C.-Y. Zhang & H. Zhang, arXiv:2411.03193]

- $H(Z)$ should be a continuous real function decreasing monotonically from 1 to -1 in the range $[Z_c, Z_h]$, with a zero at Z_0.

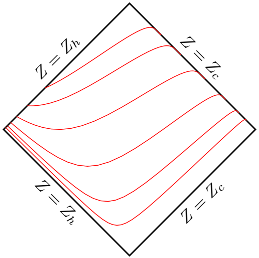

Figure: Penrose diagram demonstrating the hyperboloidal slices of the RN-dS spacetime, which penetrate both the outer horizon of the black hole and the cosmological horizon

Linear stability analysis

- The static background RN-dS black hole has been analytically obtained under the hyperboloidal coordinates.

Linear stability analysis

- The static background RN-dS black hole has been analytically obtained under the hyperboloidal coordinates.
- Computing the QNM on top of this background to reproduce the linear (in)stability under the hyperboloidal coordinates
  - Traditional methods for computing QNM (too many to list here)
  - Linear perturbations and QNM under the hyperboloidal coordinates in the asymptotically flat case through the height function technique [A. Zenginoğlu, arXiv:1102.2451]
  - Our choice: the linearized evolution scheme to deal with the nonlinear dynamics and QNM in a unified framework [R. Li, Y. Tian, H. Zhang & J. Zhao, arXiv:1506.04267; Y. Du, S. Lan, YT & H. Zhang, arXiv:1511.07179]

Linear stability analysis

- The linearized evolution scheme for QNM:
  - Gauge is fixed in the same way as in the nonlinear dynamic evolution.
  - The equations of motion are linearized, which govern the dynamics of linear perturbations, on top of analytical or numerical backgrounds.
  - Separation of variables is possible if the background has symmetries.
  - The linearized dynamics is either: 1) evolved in real time and then Prony or 2) reformulated as a generalized eigenvalue problem

$$(\boldsymbol{A} - \omega \boldsymbol{B})P = 0 \tag{2.1}$$

for the quasi-normal frequency ω, where the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ depend on the background and $\delta\Psi = P(Z)e^{-i\omega T}$ is the collection of all the perturbation fields.

The parameter region of instability

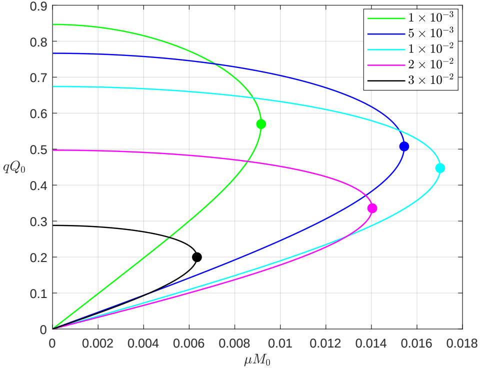

Figure: $\lambda_{\min}$ (lines below the filled dots) and $\lambda_{\max}$ (lines above the filled dots) as functions of μ (scalar mass) with $M_0 = 0.2$, $Q_0 = 0.3M_0$ and different ΛM_0^2

- Linear instability occurs if $\lambda_{\min} < qQ_0 < \lambda_{\max}$ with q the charge parameter of the scalar field.
- For a massless scalar field, unstable modes appear for arbitrarily small Λ.
- However, for massive scalar fields, there exists a lower bound of Λ, below which no instability can be observed.
- The dependence of $\lambda_{\min}$ on Q_0 is very soft, so we only show the results with fixed Q_0 here.

Outline

Simulation of black hole dynamics under the hyperboloidal coordinates

- The following gauge should work for nonlinear dynamics of spherically symmetric black holes:

$$\mathrm{d}s^2 = \frac{L^2}{Z^2}\mathrm{e}^{-\chi(T,Z)}[-\mathrm{d}T^2 + (\mathrm{d}Z + H(T,Z)\mathrm{d}T)^2] + \frac{L^4}{Z^2}\mathrm{d}\Omega^2. \tag{3.1}$$

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- As tests, it works very well in many dynamical processes of spherically symmetric black holes in the asymptotically flat case (typically more efficient and more accurate than traditional methods in long time evolution).

Lagrangian of the system

- Consider the Einstein-Maxwell-scalar-Λ (EMSΛ) system

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} + 4\mathcal{L}_m\right], \tag{3.2}$$

$$\mathcal{L}_m = -D_\mu\psi\overline{D^\mu\psi} - \mu^2\psi\bar{\psi}, \qquad D_\mu := \partial_\mu - iqA_\mu \tag{3.3}$$

- To simplify the equations of motion, we introduce the following auxiliary variables

$$B = \mathrm{e}^{\chi}A', \qquad (A := A_T, \quad A_Z = 0) \tag{3.4}$$

$$\zeta = \dot{\psi} - H\psi' - iqA\psi, \tag{3.5}$$

where dot and prime denote the derivative with respect to the temporal coordinate T and the radial coordinate Z, respectively.

Equations of motion in the hyperboloidal coordinates

A system of evolution equations with a very simple structure:

$$\dot{\chi} - 2H\chi' = 4Z\mathrm{Re}\left[\zeta\bar{\psi}'\right] \tag{3.6}$$

$$\dot{H} - 2HH' = \frac{3(1-H^2)}{Z} - Z\frac{e^{-\chi}}{L^2} + Z^3\frac{e^{-\chi}B^2}{L^2} + \frac{e^{-\chi}L^2}{Z}\left[\Lambda + 2\mu^2|\psi|^2\right] \tag{3.7}$$

$$\dot{B} - HB' = -\frac{2qL^2}{Z^2}\mathrm{Im}\left[\bar{\psi}\psi'\right] \tag{3.8}$$

$$\dot{\psi} - H\psi' = \zeta + iqA\psi \tag{3.9}$$

$$\dot{\zeta} - H\zeta' = Z^2\left[\left(\frac{\psi'}{Z^2}\right)' + \zeta\left(\frac{H}{Z^2}\right)'\right] + iqA\zeta - \mu^2\frac{L^2}{Z^2}e^{-\chi}\psi \tag{3.10}$$

Note that the pair of (3.9) and (3.10) forms a wave equation for ψ, while (3.6-3.8) are advection equations.

Equations of motion in the hyperboloidal coordinates

The constraints

$$0 = -(1-H^2)\chi' + 2Z[|\zeta|^2 + |\psi'|^2 + 2H\mathrm{Re}\left(\zeta\bar{\psi}'\right)] - Z^3(\frac{1-H^2}{Z^3})' - \frac{Ze^{-\chi}}{L^2} + \frac{Z^3e^{-\chi}B^2}{L^2} + \frac{e^{-\chi}L^2}{Z}\left[\Lambda + 2\mu^2|\psi|^2\right] \tag{3.11}$$

$$0 = B' + \frac{2qL^2}{Z^2}\mathrm{Im}\left[\bar{\psi}\zeta\right] \tag{3.12}$$

are preserved by the evolution equations, which can be used to check the numerical errors.

No artificial BC is needed at the event horizon or the cosmological horizon!

Nonlinear dynamics and the final state of the instability

We have five fields $\{H, \chi, \psi, \zeta, B\}$, whose T derivative can be calculated through the evolution equations (3.6-3.10).

- T direction: explicit fourth-fifth order Runge-Kutta marching with adaptive steps (to efficiently perform long time evolutions)
- Z direction: Chebyshev pseudo-spectral expansion

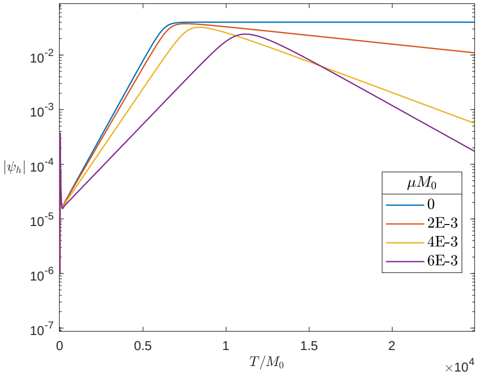
Two constraints allow us to choose the initial data of 3 dynamical variables freely:

$$\psi(T=0,Z) = k\exp[-(\frac{Z-c}{w})^2], \tag{3.13}$$

$$\zeta(T=0,Z) = -H\psi', \tag{3.14}$$

$$\chi'(T=0,Z) = 2Z|\psi'|^2. \tag{3.15}$$

Characteristic Behaviors of the evolution of scalar fields



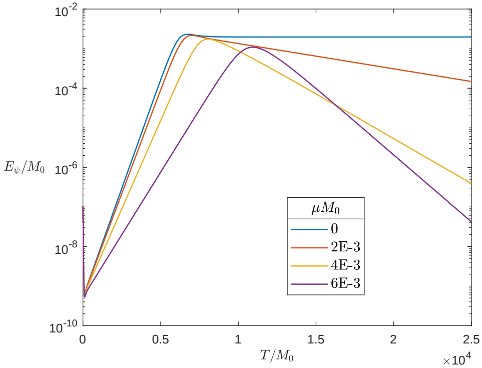


Figure: Evolutions of the scalar field $|\psi_h|$ on the apparent horizon (left) and the scalar field energy E_ψ (right) exhibit two distinct stages of the dynamics: the so-called superradiant growth stage and relaxation stage.

Discharge of the RN-dS black hole

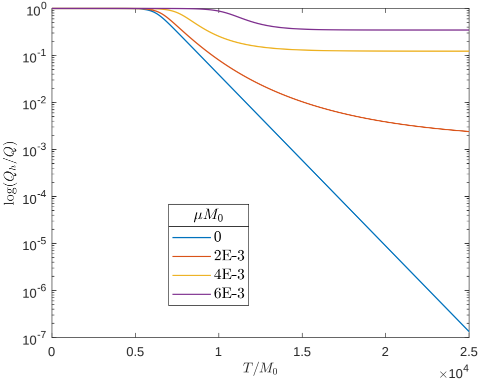


Figure: Evolutions of the black hole charge Q_h for different scalar mass μ.

- Recall that the smaller μ, the smaller $\lambda_{\min}$, and $\lambda_{\min} \equiv 0$ when $\mu = 0$.
- Note that Q_h decays exponentially when $\mu = 0$. In this case, the instablity will not settle down until all the BH charge has been extracted.
- The results show that the smaller $\lambda_{\min}$ is, the more complete the charge extraction will be.
- The final state is a bald black hole.

Numerical Errors and Convergence

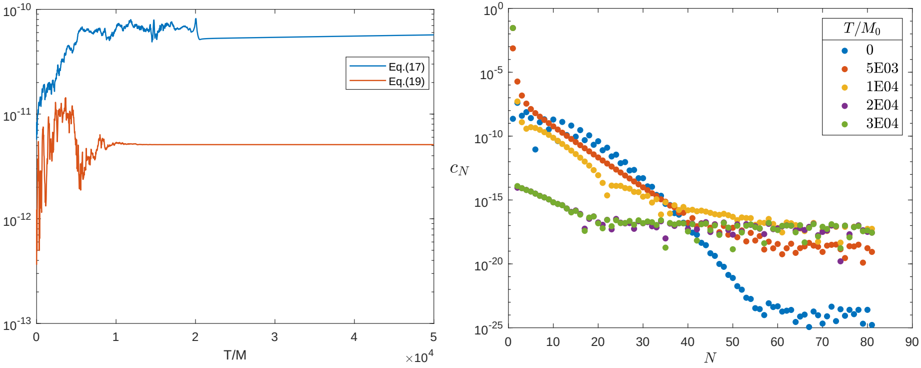


Figure: Left: The variation of the maximal Einstein and Maxwell constraint violations along the Z direction with time. Right: The snapshots of the spectral coefficients of the field χ.

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- Dynamics of scalar perturbations is governed by the linearization of the corresponding equations of motion (3.9) and (3.10).
- In the linearized evolution scheme for QNM computation, no BC is needed in the hyperboloidal coordinates.
- Time translation symmetry of the background enables the expansion with quasi-normal modes $\delta\psi = \tilde{\psi}(Z)e^{-i\omega T}$ and $\delta\zeta = \tilde{\zeta}(Z)e^{-i\omega T}$, which implies the substitution

$$\partial_T \to -i\omega$$

when plugged into the linearized equations of motion.

Supplementary: QNM computation

- The linearized equations of motion becomes

$$(\boldsymbol{A}-\omega\boldsymbol{B})\begin{pmatrix}\tilde{\zeta}\\ \tilde{\psi}\end{pmatrix}=0 \tag{3.16}$$

with differential operators

$$\boldsymbol{A}=\begin{pmatrix}1 & H\partial_Z - iqQ_0\frac{Z}{L^2}\\ \partial_Z H-\frac{2}{Z}H-iqQ_0\frac{Z}{L^2} & \partial_Z^2-\frac{2}{Z}\partial_Z-\mu^2\frac{L^2}{Z^2}\end{pmatrix},\quad \boldsymbol{B}=-i\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{3.17}$$

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  which is a generalized eigenvalue problem for ω.
- Upon discretization in Z, the quasi-normal frequencies ω can be numerically obtained as generalized eigenvalues of matrices.

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Summary and outlook

- The hyperboloidal formalism has the advantage of no need to put artificial BC when dealing with asymptotically flat or dS black holes.
- The nonlinear evolution of unstable charged de Sitter black holes can be performed with high efficiency and long term accuracy in a simple form of hyperboloidal coordinates.

Summary and outlook

- The hyperboloidal formalism has the advantage of no need to put artificial BC when dealing with asymptotically flat or dS black holes.
- The nonlinear evolution of unstable charged de Sitter black holes can be performed with high efficiency and long term accuracy in a simple form of hyperboloidal coordinates.
- It seems not easy to generalize our simple nonlinear formalism to less symmetric cases.
- Further exploration of optimized/adapted formalisms of NR for various problems is still interesting and important.

Thank You!