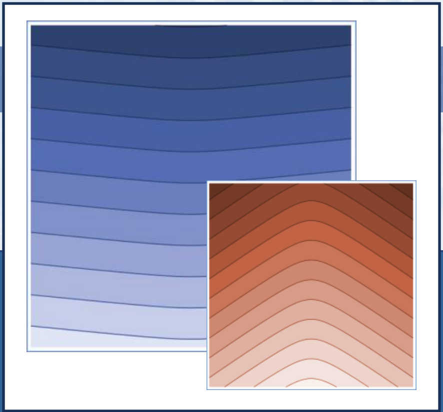
Contact at cdb23@st-andrews.ac.uk

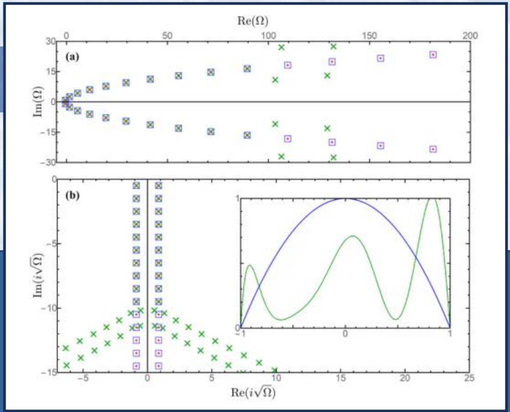
QSimFP

EPSRC
Engineering and Physical Sciences Research Council

Generalising Hyperboloidal Methods for Non-Relativistic Systems

Hyperboloidal Research Network





Christopher Burgess
University of St Andrews
17th January 2025

Based on C. Burgess, F. Koenig. Frontiers in Physics 12:1457543 (2024) & C. Burgess, *et al*. Physical Review Letters 132:053802 (2024)

Hyperboloidal Methods

for Quasinormal Mode Frequencies

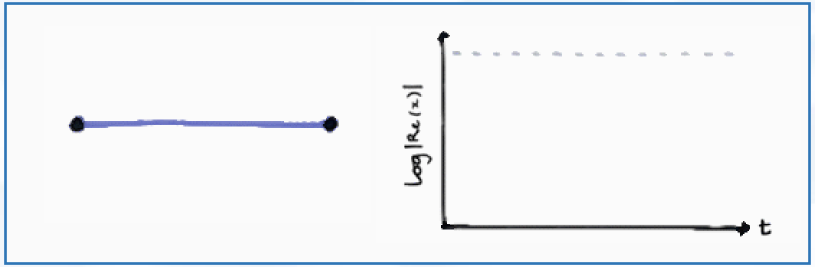
$$\omega = \omega_R + i\omega_I$$

Review	Quasinormal Modes & Wave Equation
Content	Schrödinger
Outlook	Beyond Schrödinger

Quasinormal Modes

$$\omega = \omega_R + i\omega_I$$

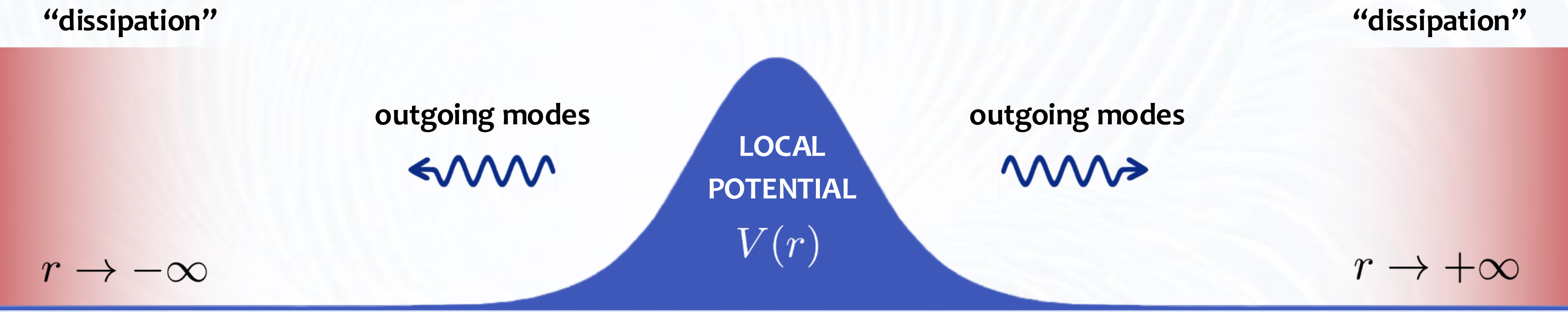
Normal modes



Separation of variables	$\phi(r,t) = u(r)T(t)$	Separation of variables	$\phi(r,t) = u(r)T(t)$
Conservative		Dissipative	
Oscillatory & Periodic	$T(t) = e^{-i\omega t}$	Oscillatory & Decaying	$T(t) = e^{-i\omega_R t}e^{\omega_I t}$
Real eigenvalues	$\omega = \omega_R$	Complex eigenvalues	$\omega = \omega_R + i\omega_I$
Spectral theorem		No spectral theorem	
Field expansion	$\phi \sim \int \mathrm{d}\omega\ A(\omega) u_\omega(r) e^{-i\omega t}$	Ringdown expansion[1]	$\phi \sim \sum a_\omega u_\omega(r) e^{-i\omega t}$

[1] K. Kokkotas, B. Schmidt. Living Reviews in Relativity 2:2 (1999)

How can a conservative wave equation have QNMs?



Wave Equation

$$\left(\partial_t^2 - \partial_r^2 + V\right)\phi = 0$$

Wave Equation

$$\left(\partial_t^2 - \partial_r^2 + V\right)\phi = 0$$

Mode Equation

$$\left(-\omega^2 - \partial_r^2 + V\right)u = 0 \quad \textbf{with} \quad \phi = u e^{-i\omega t}$$

Boundary Conditions[2]

$$\phi \sim e^{ikr - i\omega t} \quad \textbf{as} \quad r \to \pm\infty \quad \textbf{with} \quad k = \pm\omega$$

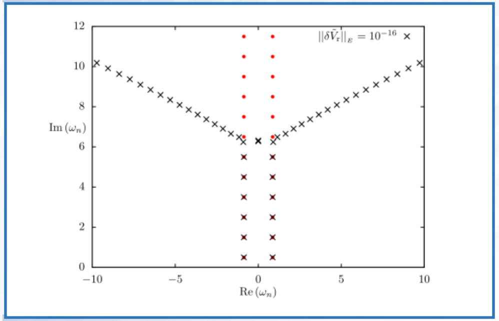
$$v = \omega / k$$

Compactified Hyperboloidal Method for Quasinormal Modes[3,4]

- **I. Coordinate Transformation** $t = \tau - h(x) \qquad r = g(x)$
- **II. Reduction in Time** $\psi = \partial_\tau \phi$
- **III. Eigenvalue Problem** $Lu = \omega u$
- **IV. Discretization** $L \to L^N \qquad u \to u^N$
- **V. Numerical Solver** $\omega \in \{1.00 - 0.50i, 1.00 - 1.50i, \ \ldots\}$

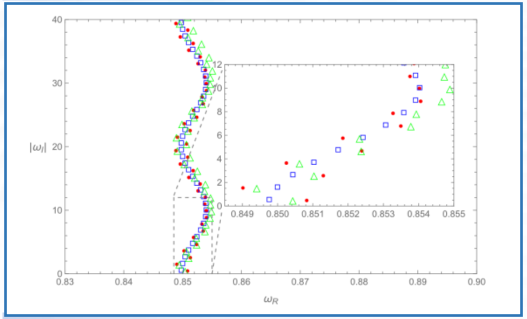
[3] A. Zenginoğlu. Journal of Computational Physics 230:2286 (2011)
[4] J. Jaramillo, *et al.* Physical Review X 11:031003 (2021)

Pöschl-Teller Potential



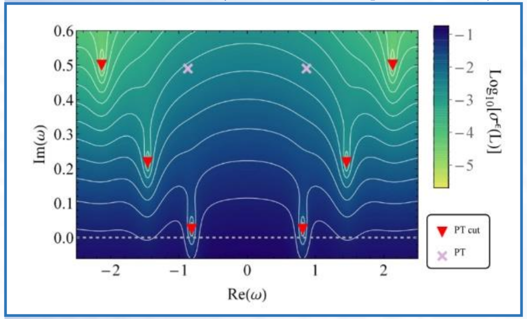
Jaramillo *et al.* PHYS. REV. X **11**, 031003 (2021)

Modified Pöschl-Teller



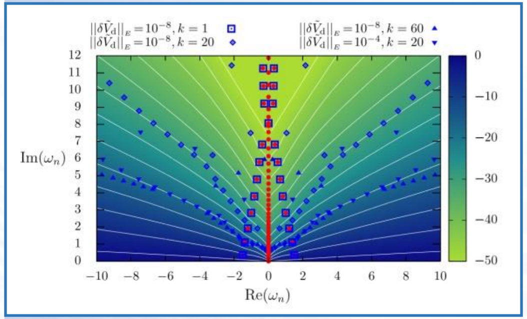
Li *et al.* PHYS. REV. D **110**, 064076 (2024)

Pöschl-Teller (& Pseudospectrum)



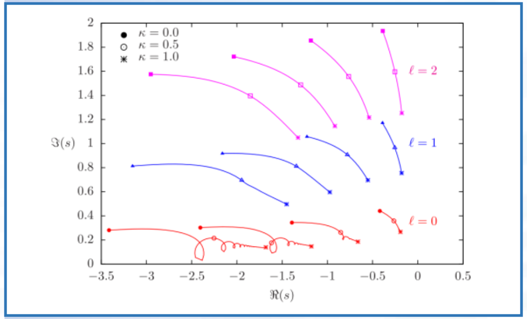
Boyanov *et al.* PHYS. REV. D **107**, 064012 (2023)

Schwarzschild BH



Jaramillo *et al.* PHYS. REV. X **11**, 031003 (2021)

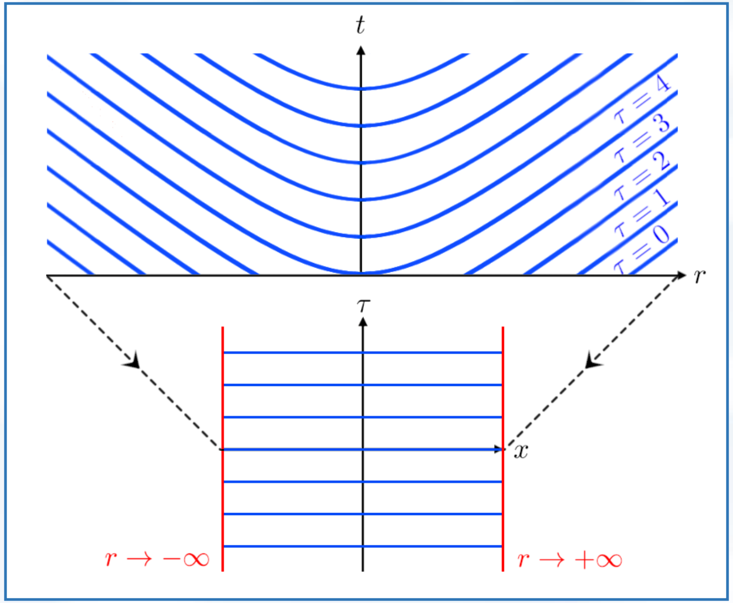
Reissner-Nordström BH



Macedo *et al.* PHYS. REV. D **98**, 124005 (2018)

I. Coordinate Transformation

$$t = \tau - h(x) \qquad r = g(x)$$



τ : *asymptotic contours of constant field*

Partial Derivatives

$$\partial_t = \partial_\tau \qquad \partial_r = \frac{\partial_x h}{\partial_x g}\partial_\tau + \frac{1}{\partial_x g}\partial_x$$

QNM Solutions

→ **finite everywhere on the space**

II. Reduction in Time

$$\psi = \partial_\tau \phi$$

$\longrightarrow$ $$\partial_\tau \psi = L_1 \phi + L_2 \psi$$

spatial operators L_1, L_2

III. Eigenvalue Problem

$$Lu = \omega u$$

$$u = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \qquad L = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$$

$\longrightarrow$ self-adjoint in the bulk
non-selfadjoint on the boundary

IV. Discretization

$L \to L^N \quad u \to u^N$

$$Lu = \omega u \;\Rightarrow\; L^N u^N = \omega u^N$$

polynomial interpolation

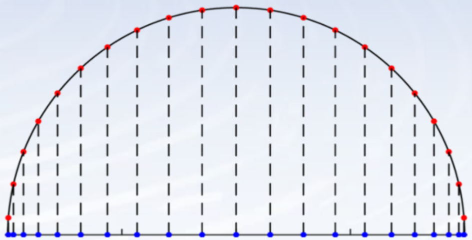
$$u_{\text{int}}(x) = \sum_{j=0}^{N-1} u(x_j) \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}$$

derivatives $\partial_x \to D^N$

$$D^N_{ij} = \frac{\sum_{l \neq j} \prod_{k \neq j,l} (x_i - x_k)}{\prod_{k \neq j} (x_j - x_k)}$$

Chebyshev Extremal Points

$$x_j = \cos(j\pi/(N-1))$$



V. Numerical Solver

Schrödinger

$$(i\partial_t - \partial_r^2 + V)\phi = 0$$

Quantum Mechanics

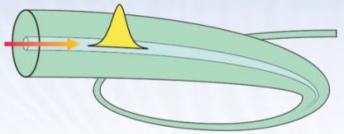
QNMs of "quantum" equation

Generalized Wave Equations

higher-derivative theories: $\sim (i\partial_r)^k$

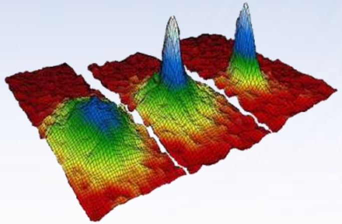
Analogue Gravity

light in optical fibres[5]



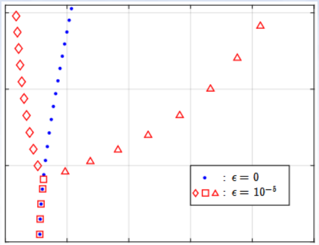
$$\left(\partial_z - i\beta(i\partial_\tau + \omega_s) - \beta(\omega_s) - i\gamma|A|^2\right)A = 0$$

Bose-Einstein condensates[6]



$$\left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\psi|^2\right)\psi = 0$$

BH spectral instability[7]



$$V = V_0 + \epsilon\Delta V$$

[5] C. Burgess, *et al.* Physical Review Letters 132:053802 (2024)
[6] E. Gross. Il Nuovo Cimento 20:454 (2007)
[7] T. Torres. Physical Review Letters 131:111401 (2023)

Schrödinger Equation

$$(i\partial_t - \partial_r^2 + V)\phi = 0$$

Mode Equation

$$\left(\omega - \partial_r^2 + V\right) u = 0 \quad \textbf{with} \quad \phi = u e^{-i\omega t}$$

Boundary Conditions

$$\phi \sim e^{ikr - i\omega t} \quad \textbf{as} \quad r \to \pm\infty \quad \textbf{with} \quad \operatorname{sgn}(j) \to \pm 1$$

("outgoing")

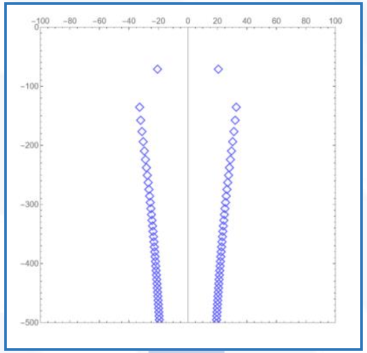
Continuity Equation

$$\partial_t \underbrace{(\phi\phi^*)}_{\text{density } \rho} + \partial_r \left(\underbrace{i(\phi\partial_r\phi^* - \phi^*\partial_r\phi)}_{\text{current } j} \right) = 0$$

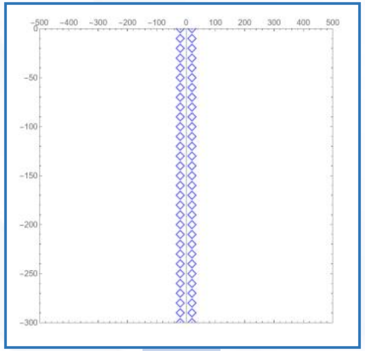
Relativistic

e.o.m $$\left(\partial_t^2 - \partial_r^2 + V\right)\phi = 0$$

mode $$\left(-\omega^2 - \partial_r^2 + V\right)u = 0$$



V_A



V_B

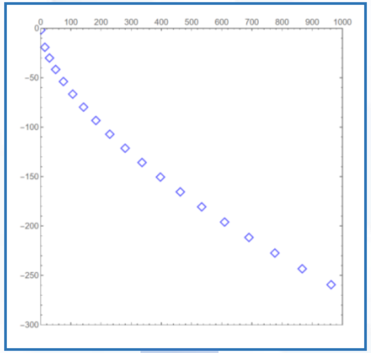
$$-\omega^2 \leftrightarrow \omega$$

mode analogy

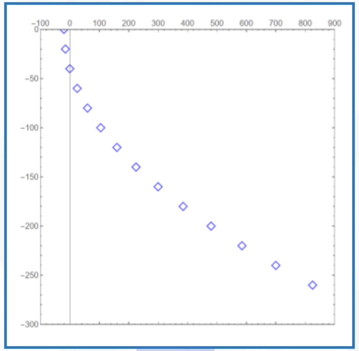
Non-relativistic

e.o.m $$\left(i\partial_t - \partial_r^2 + V\right)\phi = 0$$

mode $$\left(\omega - \partial_r^2 + V\right)u = 0$$



V_A



V_B

Non-Relativistic Compactified Hyperboloidal Method for Quasinormal Modes[8]

- **I. Coordinate Transformation** $t = \tau - h(x) \qquad r = g(x)$
- **II. Reduction in Time** $\psi = \partial_\tau \phi$
- **III. Eigenvalue Problem** $Lu = \omega u$
- **IV. Discretization** $L \to L^N \quad u \to u^N$
- **V. Numerical Solver** $\omega \in \{1.00 - 0.50i, 1.00 - 1.50i, \ \ldots\}$

[8] C. Burgess, F. Koenig. Frontiers in Physics 12:1457543 (2024)

I. Coordinate Transformation

$$t = \tau - h(x) \qquad r = g(x) \longrightarrow \left(-i\partial_\tau + \left[\frac{\partial_x h}{\partial_x g} \partial_\tau + \frac{1}{\partial_x g} \partial_x \right]^2 - V \right) \phi = 0$$

II. Reduction in Time

$$\psi = \partial_\tau \phi \longrightarrow \partial_\tau \psi = L_1 \phi + L_2 \psi$$

spatial operators L_1, L_2

III. Eigenvalue Problem

$$Lu = \omega u \qquad u = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \qquad L = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$$

IV. Discretization

V. Numerical Solver

→ **spurious results**

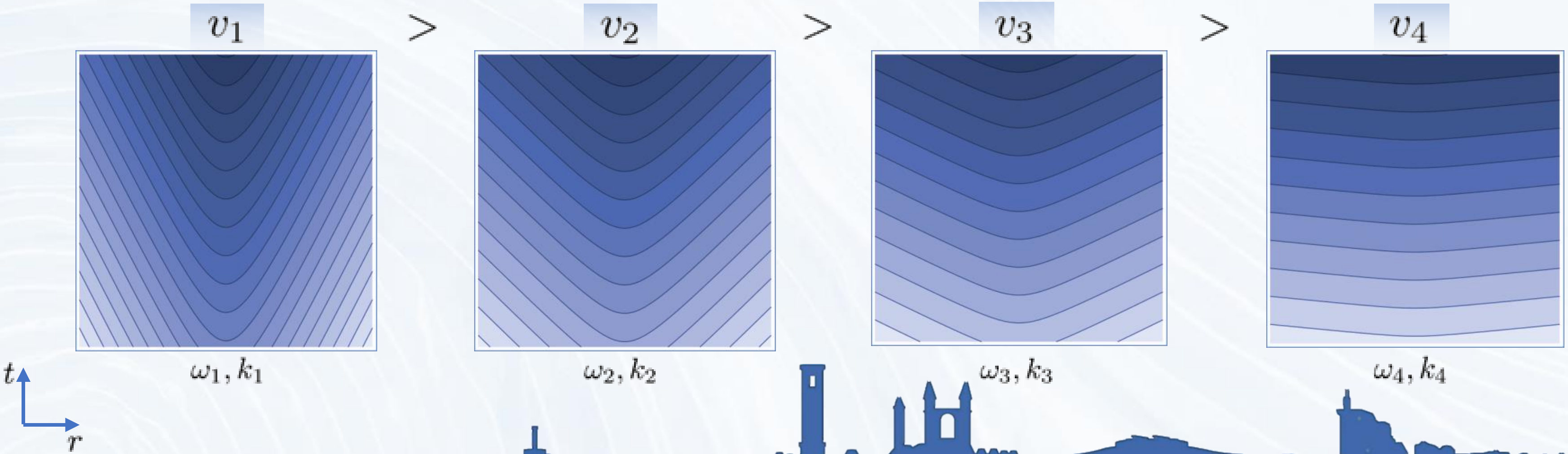
Non-Relativistic Dispersion Relation

$$\omega = -k^2 \qquad v_g \sim \frac{\mathrm{Im}(\omega)}{\mathrm{Im}(k)}$$

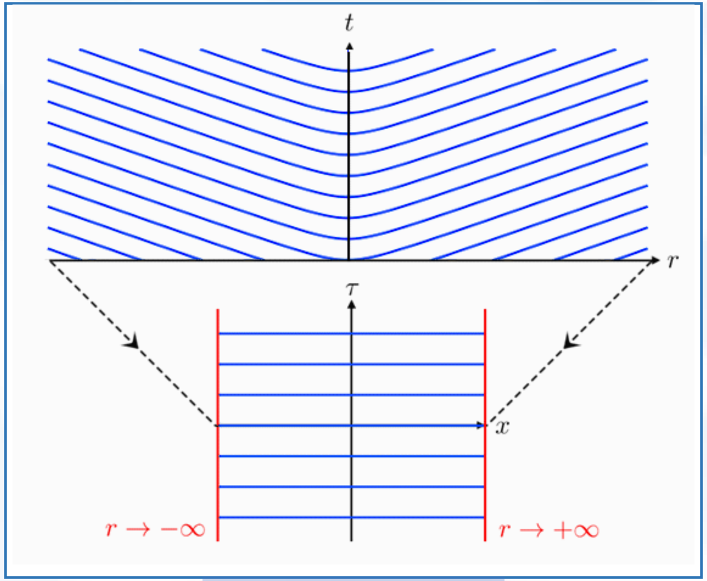
No unique coordinate transformation

$$t = \tau - h(x) \qquad r = g(x)$$

Contours of Constant Amplitude



I. Parametrized Coordinate Transformation $t = \tau - h(x) \quad r = g(x)$



varying v_g

Parametrized Height Function

$$h(x) = v_g^{-1} h_0(x) \qquad \frac{\Delta r}{\Delta t} \to \pm v_g$$

τ : *asymptotic contours of constant amplitude*
for modes of a given frequency

Compactification Function

$$g(x) = g_0(x)$$

amplitude finite everywhere

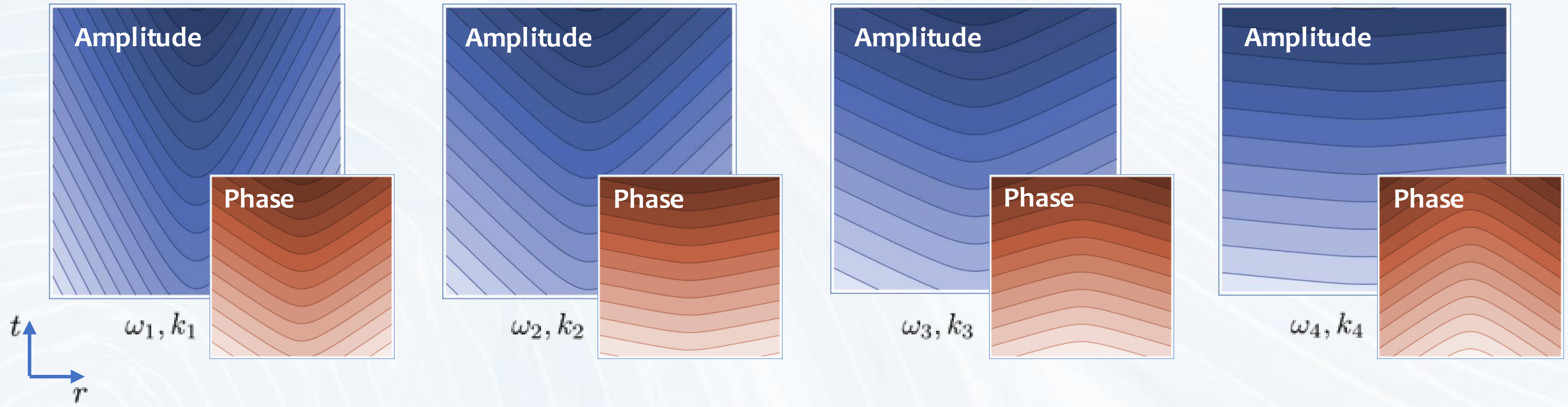
Dispersion Relation

$$\omega = -k^2 \qquad v_g \sim \frac{\mathrm{Im}(\omega)}{\mathrm{Im}(k)} \qquad v_p \sim \frac{\mathrm{Re}(\omega)}{\mathrm{Re}(k)}$$

Phase Singularities

at $x = \pm 1$

Contours of Constant Amplitude & Phase



II. Parametrized Phase Rotation

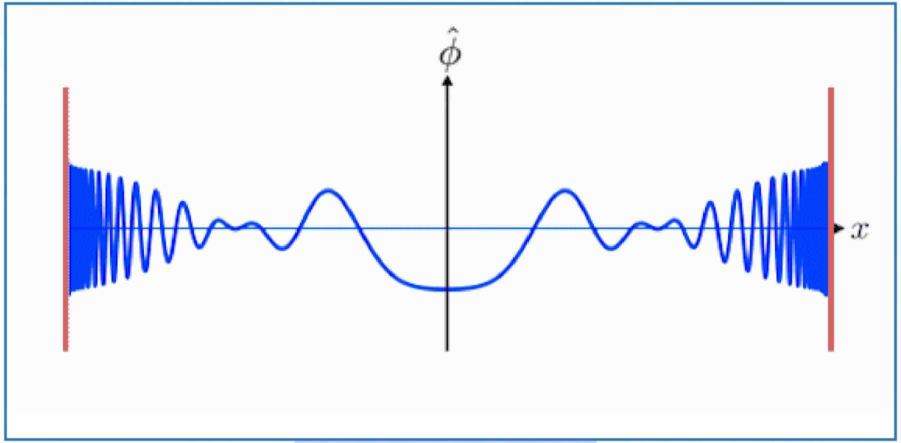
phase rotation removes phase singularity

for a given group/phase velocity mismatch

$$\hat{\phi} = e^{-i\Delta h_0(x)}\phi$$

with $\Delta = \frac{1}{2}(v_g - v_p)$ group/phase velocity mismatch

amplitude & phase finite everywhere



varying Δ

III. Reduction in Time

$$\hat{\psi} = \partial_\tau \hat{\phi}$$

$$x^2 \partial_\tau \hat{\psi} = J_1 \hat{\phi} + J_2 \hat{\psi}$$

parametrized by v_g, v_p

Example: *Pöschl-Teller Potential*

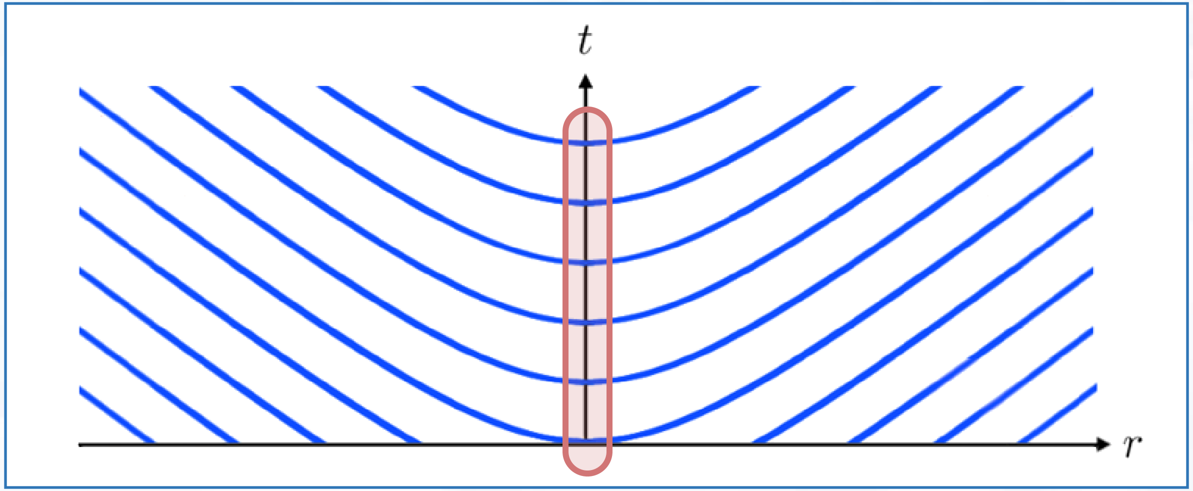
$$J_1 = v_g^2 \left[V - i\Delta(1-x^2) + \Delta^2 x^2 + 2(1-i\Delta)x(1-x^2)\partial_x - (1-x^2)^2 \partial_x^2 \right]$$
$$J_2 = v_g \left[i v_g + 1 - x^2 + 2i\Delta x^2 + 2x(1-x^2)\partial_x \right]$$

$$x^2\partial_\tau\hat{\psi} = J_1\hat{\phi} + J_2\hat{\psi} \xrightarrow{?} \partial_\tau\hat{\psi} = L_1\hat{\phi} + L_2\hat{\psi}$$

$$\sim \left(\frac{\partial_x h}{\partial_x g}\right)^2$$

vanishes

at $x = r = 0$



IV. Power Series Construction

$$\partial_\tau \hat{\psi} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \partial_\tau \hat{\psi}^{(k)}\Big|_{x=0}$$

Repeated differentiation of e.o.m. yields $\partial_\tau \hat{\psi}^{(k)}\big|_{x=0}$

Separation of Terms

$$\hat{\psi} = \hat{\psi}_a + \hat{\psi}_b$$

$$x^2\partial_\tau\hat{\psi} = x^2(L_1^a\hat{\phi}_a + L_2^a\hat{\psi}_a) + J_1^b\hat{\phi}_b + J_2^b\hat{\psi}_b$$

divisible by x^2 — indivisible by x^2

$$\partial_\tau\hat{\psi}_a = L_1^a\hat{\phi}_a + L_2^a\hat{\psi}_a$$

$$x^2\partial_\tau\hat{\psi}_b = J_1^b\hat{\phi}_b + J_2^b\hat{\psi}_b$$

Equation of Motion

$$x^2\partial_\tau\hat{\psi} = J_1\hat{\phi} + J_2\hat{\psi} \;\longrightarrow\; \partial_\tau\hat{\psi} = L_1\hat{\phi} + L_2\hat{\psi}$$

V. Implicit Eigenvalue Equation

$$L_\omega u = \omega u$$

$$L = L_\omega$$ parametrized by $v_g, v_p \sim \omega_R, \omega_I$

$$u \equiv \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \end{pmatrix}, \quad L \equiv i \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$$

Non-relativistic Dispersion Relation

$$\omega = -\frac{1}{2} v_g \left(v_p + i\sqrt{v_g(v_g - 2v_p)} \right)$$

uniquely defines quasinormal mode frequencies

VI. Discretization

$$L \to L^N \quad u \to u^N$$

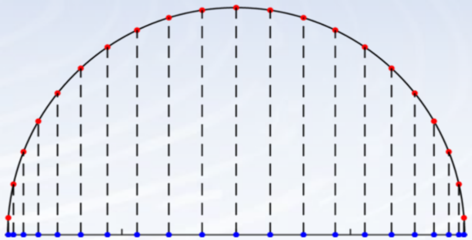
$$Lu = \omega u \Rightarrow L^N u^N = \omega u^N$$

$$\det\left(L^N - \omega \mathrm{Id}\right) = 0$$

solve for QNM frequencies

Chebyshev Extremal Points

$$x_j = \cos(j\pi/(N-1))$$



discretization truncates power series!

$$\partial_x^N \to \left(D^N\right)^N = 0$$

VII. Numerical Solver

$$\underbrace{\det\left(L^N - \omega \mathrm{Id}\right)}_{2N \times 2N} = \underbrace{\det\left(\omega^2 \mathrm{Id} - i\omega L_2^N - L_1^N\right)}_{N \times N} = 0$$

with $M = \omega^2 \mathrm{Id} - i\omega L_2^N - L_1^N$.

Example: *Pöschl-Teller Potential*

$$M = \tilde{V} - \sqrt{\omega}(\sqrt{\omega}+1)\mathrm{Id} + 2(\sqrt{\omega}+1)X^N D^N + \left(2-(X^N)^2\right)\left(D^N\right)^2 + \sum_{k=0}^{N-2} \frac{\left(X^N\right)^k}{(k+2)!}\Delta^N\left[(k+2)V_1\left(D^N\right)^{k+1} + \left(V_0 + (\sqrt{\omega}+2(k+2))(\sqrt{\omega}+1)\right)\left(D^N\right)^{k+2} - \left(D^N\right)^{k+4}\right]$$

$$M = \begin{pmatrix} 6.000 + 0.9428\sqrt{\omega} + 0.6667\,\omega & -14.00 - 6.600\sqrt{\omega} - 4.000\,\omega & 10.00 + 4.714\sqrt{\omega} + 1.333\,\omega & -3.000 - 0.4714\sqrt{\omega} \\ 10.50 + 6.600\sqrt{\omega} + 2.000\,\omega & -18.00 - 12.26\sqrt{\omega} - 4.667\,\omega & 7.000 + 3.300\sqrt{\omega} & -0.5000 + 0.9428\sqrt{\omega} + 0.6667\,\omega \\ -0.5000 + 0.9428\sqrt{\omega} + 0.6667\,\omega & 7.000 + 3.300\sqrt{\omega} & -18.00 - 12.26\sqrt{\omega} - 4.667\,\omega & 10.50 + 6.600\sqrt{\omega} + 2.000\,\omega \\ -3.000 - 0.4714\sqrt{\omega} & 10.00 + 4.714\sqrt{\omega} + 1.333\,\omega & -14.00 - 6.600\sqrt{\omega} - 4.000\,\omega & 6.000 + 0.9428\sqrt{\omega} + 0.6667\,\omega \end{pmatrix}$$

Non-Relativistic Compactified Hyperboloidal Method for Quasinormal Modes

I. Parametrized Coordinate Transformation $t = \tau - v_g^{-1} h_0(x) \qquad r = g(x)$

II. Parametrized Phase Rotation $\hat{\phi} = e^{-i\Delta h_0(x)} \phi$

III. Reduction in Time $\hat{\psi} = \partial_\tau \hat{\phi}$

IV. Power Series Construction $\partial_\tau \hat{\psi} = \sum (x^k / k!) \partial_\tau \hat{\psi}^{(k)}_{x=0}$

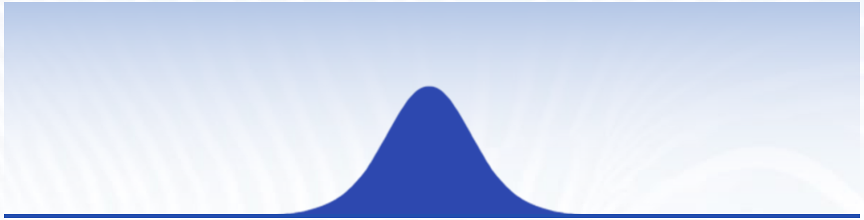
V. Implicit Eigenvalue Problem $L_\omega u = \omega u$

VI. Discretization $L \to L^N \quad u \to u^N$

VII. Numerical Solver $\omega \in \{1.00 - 0.50i, 1.00 - 1.50i, \ \ldots\}$

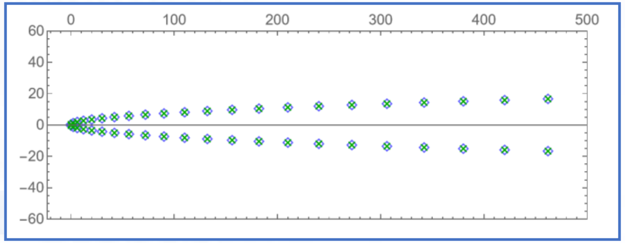
Potential

$$V = V_0 \,\mathrm{sech}^2(r)$$

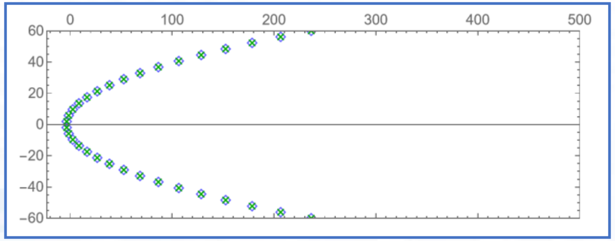


Results

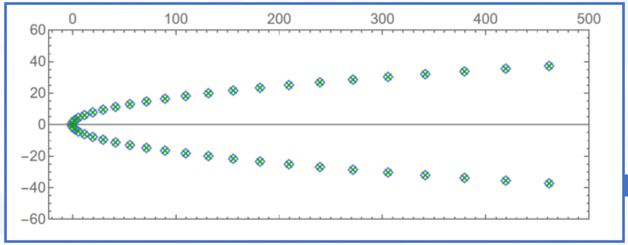
◇ Calc.
✕ Exact



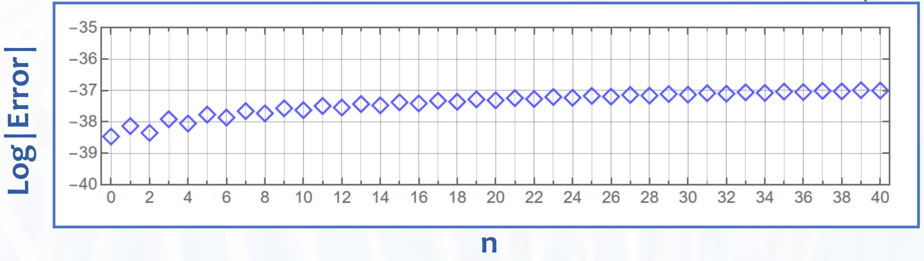
$V_0 = 0.4$



$V_0 = 4.0$

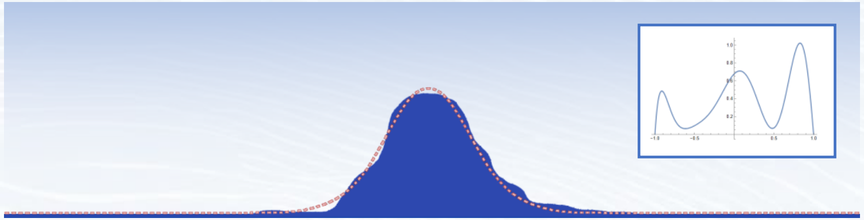


$V_0 = 1.0$



Potential

$$V = V_0 \operatorname{sech}^2(r) + \epsilon \Delta V$$

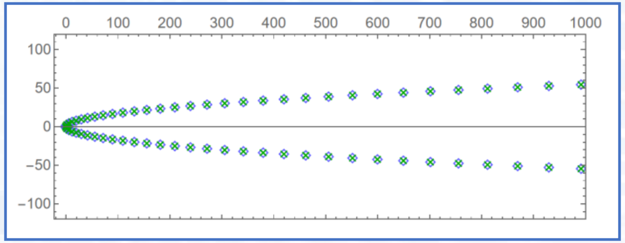


Results

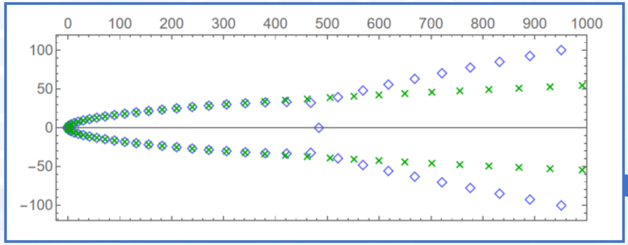
◇ Calc.

✕ P-T

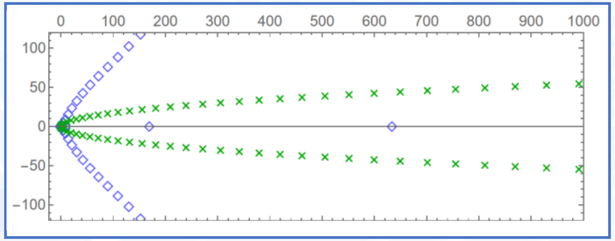
$\epsilon = 10^{-10}$

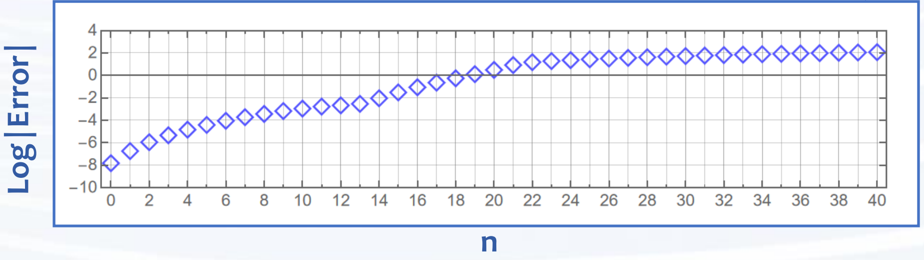


$\epsilon = 10^{-8}$



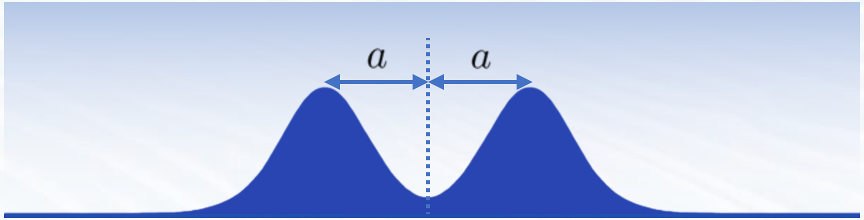
$\epsilon = 10^{-2}$



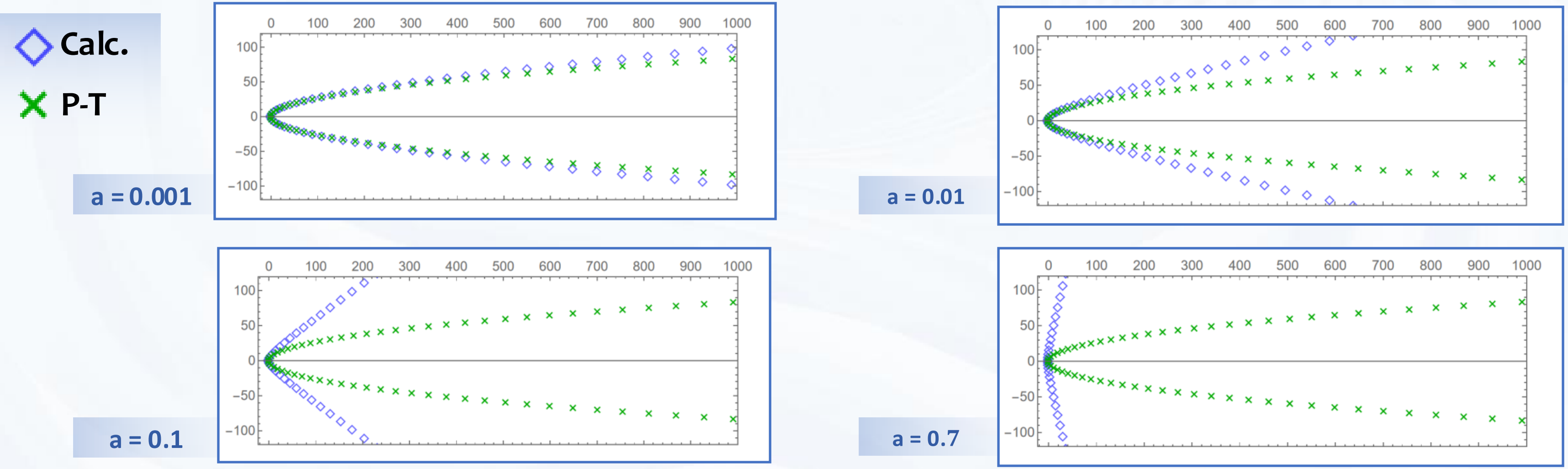


Potential

$$V = V_0\big(\mathrm{sech}(r-a) + \mathrm{sech}(r+a)\big)^2$$



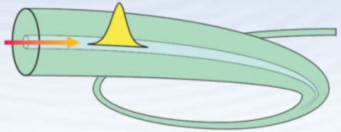
Results



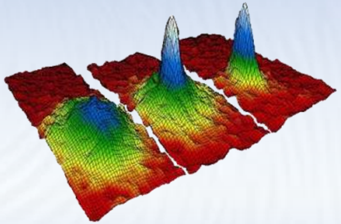
Beyond Schrödinger

$$\left(\sum_{j=1}^{n} a_j (i\partial_t)^j - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V \right) \phi = 0$$

Light in optical fibres

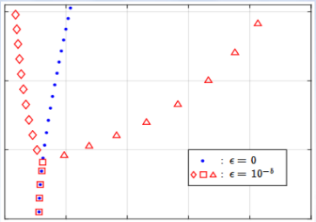


Bose-Einstein condensates



beyond non-dispersive approximation

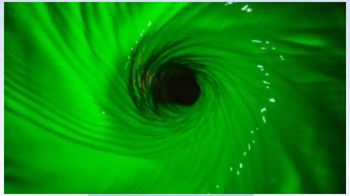
BH spectral instability



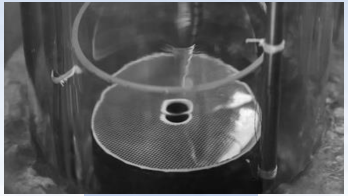
(un)stable to weak dispersion?

Lorentz violation in quantum gravity

Hydrodynamic vortex flows[9,10]



classical



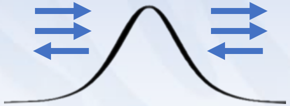
quantum

Generalization of the QNM concept

Rich phenomenology

New questions

Possibility of new insights



[9] S. Patrick, *et al.* Physical Review Letters 121:061101 (2018)
[10] P. Švančara *et al.* Nature 628:66 (2024)

Generalized Wave Equation

$$\left(\sum_{j=1}^{n} a_j (i\partial_t)^j - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V\right)\phi = 0$$

Generalized Schrödinger Equation

$$\left(i\partial_t - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V\right)\phi = 0$$

Corresponding Mode Equation

$$\left(\sum_{j=1}^{n} a_j \omega^j - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V\right)u = 0$$

Corresponding Mode Equation

$$\left(\omega - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V\right)u = 0$$

$$\sum_{j=1}^{n} a_j \omega^j \leftrightarrow \omega$$

$$\sum_{j=1}^{n} a_j (i\partial_t)^j \leftrightarrow i\partial_t$$

Generalized Schrödinger Equation

$$\left(i\partial_t - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V \right) \phi = 0$$

Mode Equation

$$\left(\omega - \sum_{j=1}^{m} b_j (-i\partial_r)^j - V \right) u = 0 \quad \text{with} \quad \phi = u e^{-i\omega t}$$

Boundary Conditions

$$\phi \sim e^{ikr - i\omega t} \quad \text{as} \quad r \to \pm\infty \quad \text{with} \quad \mathrm{sgn}(j) \to \pm 1$$

Generalized Continuity Equation

$$\partial_t(\psi\psi^*) + \partial_x \left[\sum_{j=1}^{m} b_j \sum_{k=0}^{j-1} \left((-i\partial_x)^k \psi \right) \left((i\partial_x)^{j-1-k} \psi^* \right) \right] = 0$$

density ρ

current j

Asymptotic Dispersion Relation

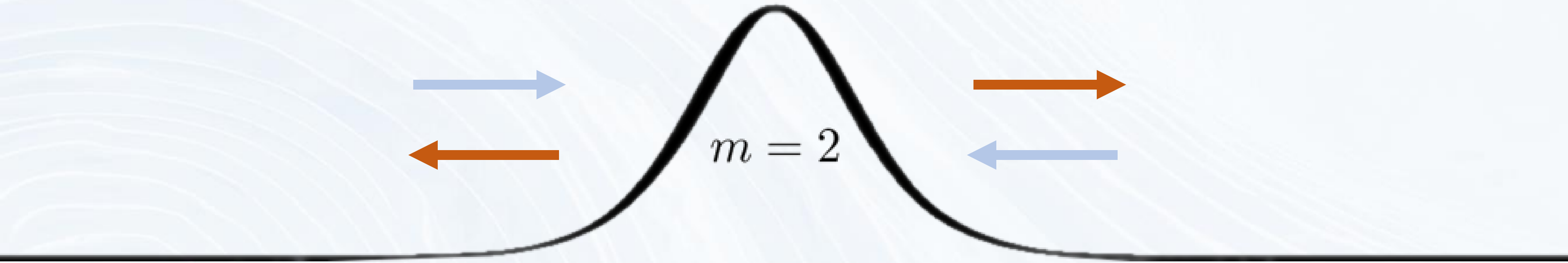
$$\omega = \sum_{j=1}^{m} b_j k^j \quad \Rightarrow \quad m \ \textbf{modes}$$

Possible Cases

left	right
1	1

$$j = j(k)$$

Example



Asymptotic Dispersion Relation

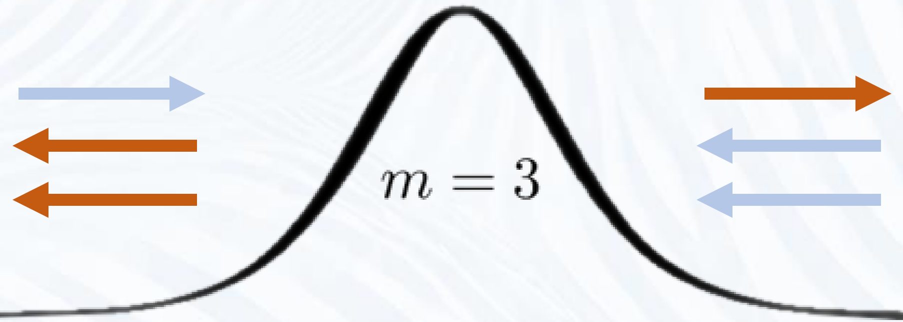
$$\omega = \sum_{j=1}^{m} b_j k^j \quad \Rightarrow \quad m \ \text{modes}$$

Possible Cases

left	right
2	1
1	2

$$j = j(k)$$

Example



Asymptotic Dispersion Relation

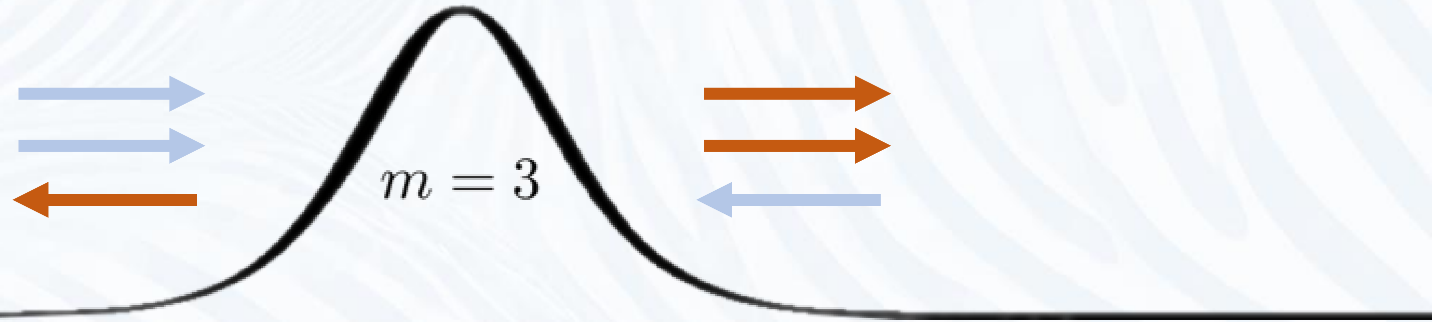
$$\omega = \sum_{j=1}^{m} b_j k^j \quad \Longrightarrow \quad m \ \ \textbf{modes}$$

Possible Cases

left	right
2	1
1	2

$$j = j(k)$$

Example



Asymptotic Dispersion Relation

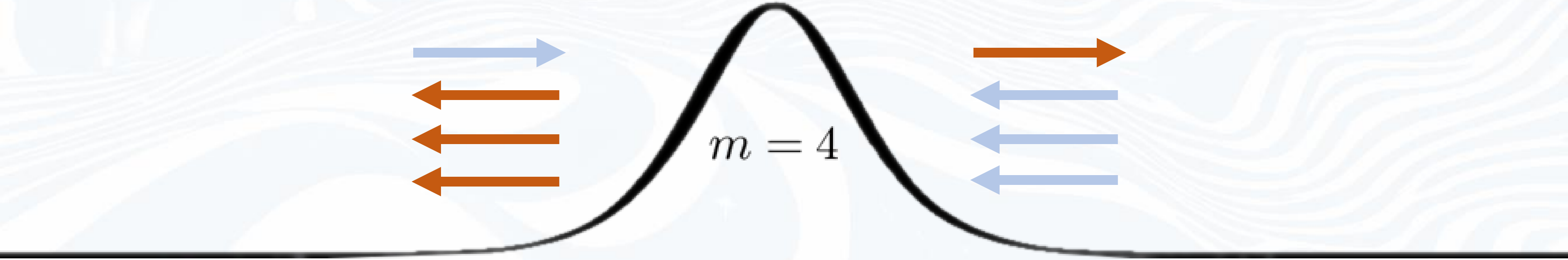
$$\omega = \sum_{j=1}^{m} b_j k^j \;\Rightarrow\; m \text{ modes}$$

Possible Cases

left	right
3	1
2	2
1	3

$$j = j(k)$$

Example



Asymptotic Dispersion Relation

$$\omega = \sum_{j=1}^{m} b_j k^j \quad \Rightarrow \quad m \text{ modes}$$

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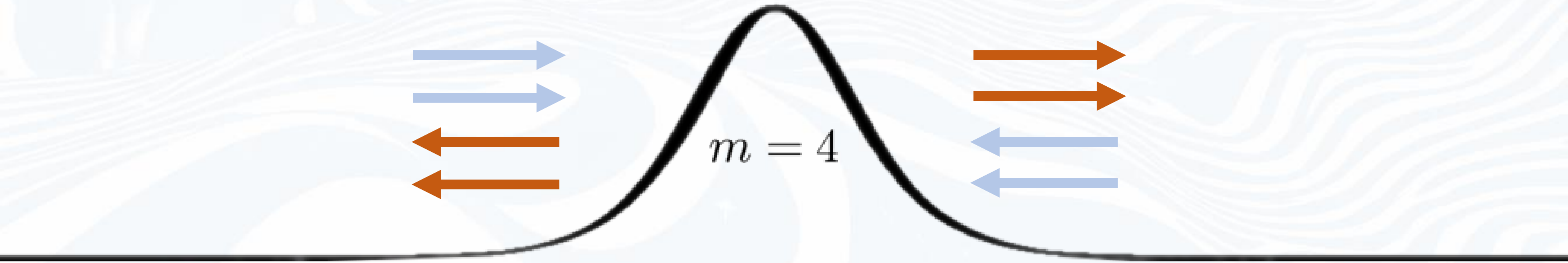
Example



Asymptotic Dispersion Relation

$$\omega = \sum_{j=1}^{m} b_j k^j \quad \longrightarrow \quad m \ \textbf{modes}$$

Possible Cases

left	right
3	1
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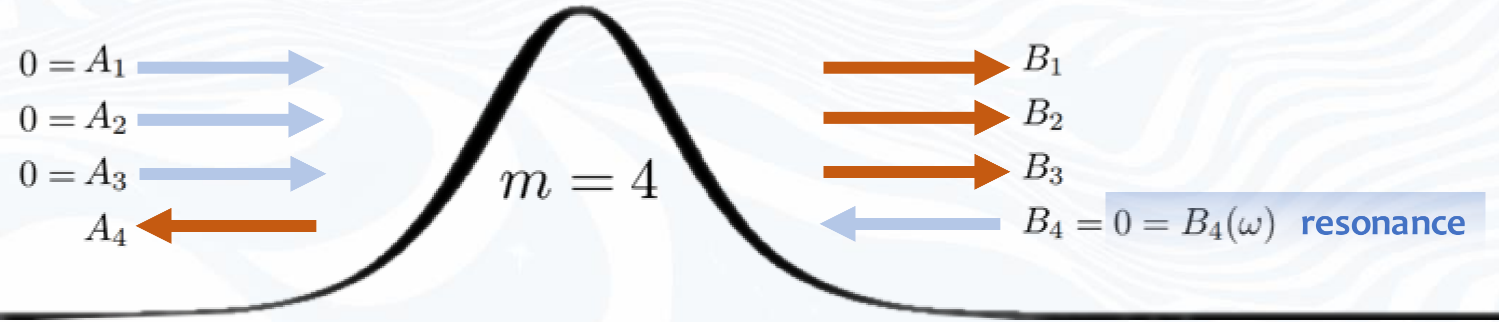
$$j = j(k)$$

Example



Asymptotic Dispersion Relation

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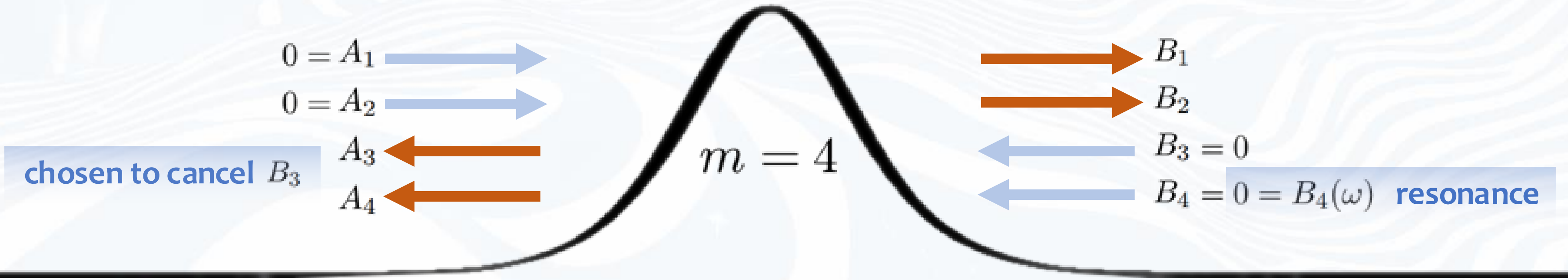
Example



Resonance Condition

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} 0 \\ A \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

Transfer Matrix

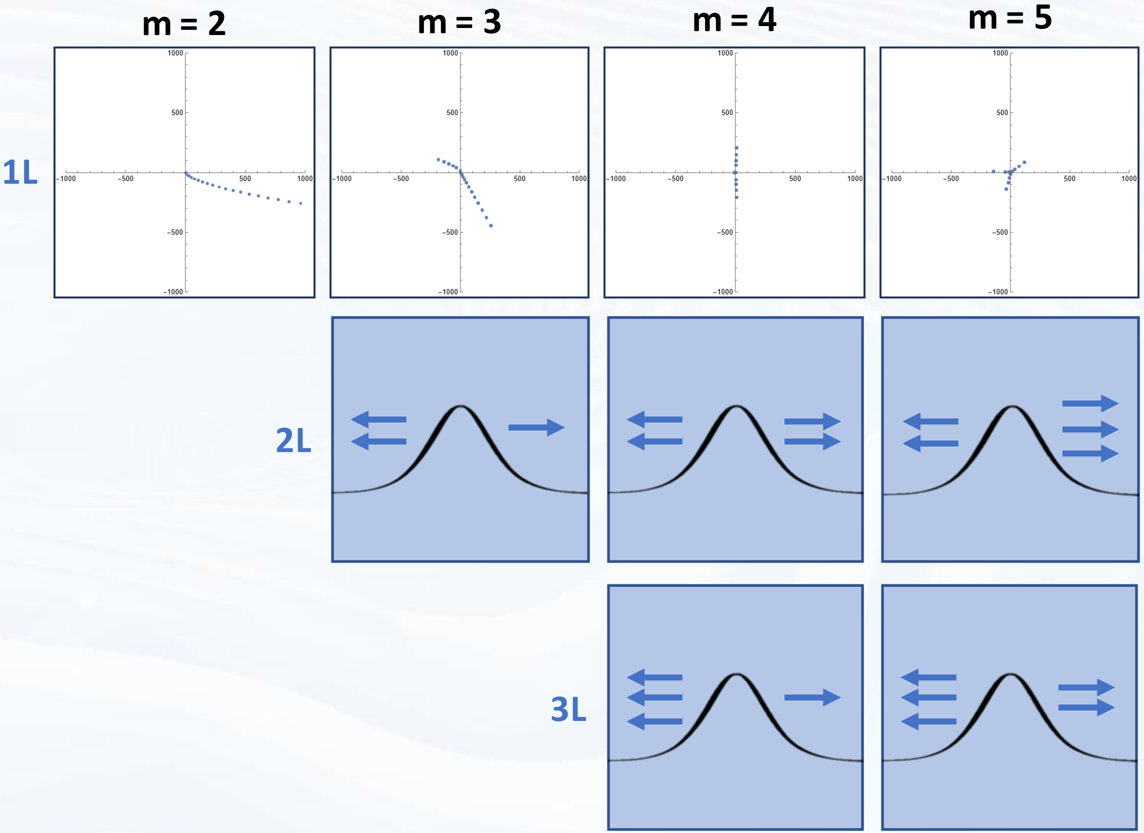
$$\det(T_4) = 0$$

Monomial Dispersion

$$\Big(i\partial_t - b_m(-i\partial_r)^m - V\Big)\phi = 0$$

Box Potential

$$V = \begin{cases} V_0 & |x| < |x_0| \\ 0 & \text{otherwise} \end{cases}$$



Summary

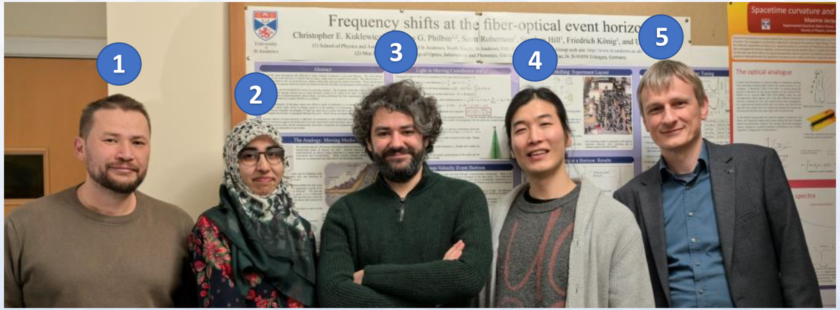
Schrödinger QNMs

- **Generalization** of QNM concept to the Schrödinger equation
- **Direct application** of hyperboloidal method to Schrödinger QNMs
- **First steps** to hyperboloidal method for QNMs of higher-order equations

Generalized Wave Equation QNMs

- **Generalization** of QNM concept to higher-order equations
- **New phenomenology** in QNMs of higher-order equations

St Andrews Team



1 Pavlos Manousiadis
2 Andleeb Zahra
1 Christopher Burgess
4 Sang-shin Baak
5 Friedrich König

Relevant papers

C. Burgess, F. Koenig. Frontiers in Physics 12:1457543 (2024)
C. Burgess, *et al.* Physical Review Letters 132:053802 (2024)