Quasi-normal mode expansion of black hole perturbation: a hyperboloidal Keldysh approach

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2412.02793 with JL. Jaramillo[1]

ongoing work with P. Bizoń*, V. Boyanov, JL. Jaramillo*, D. Pook-Kolb*

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Introduction

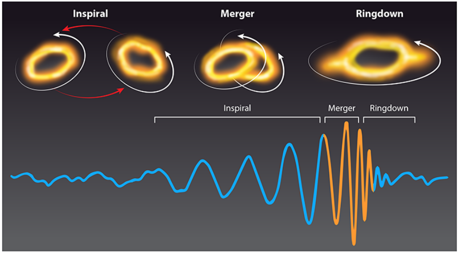

Figure: Illustration of a pair of coalescing black holes (credit : (Top) Kip Thorne; (Bottom) B. P. Abbott et al; adapted by APS/Carin Cain)

Ringdown

Sum of damped oscillators

$$\Psi_{\ell,m}(t) \sim \sum_n A_{\ell,m,n} e^{i\omega_{\ell,m,n} t}$$

Introduction : the conservative case

The conservative case

Example of system : guitar string struck
Consider the linear equation

$$\begin{cases} \partial_t u = iHu \\ u(t=0,x) = u_0(x) \end{cases}$$

where H is self-adjoint and the eigenfunctions $\hat{v}_n$ form an orthonormal basis of the Hilbert space. The solution can be written as a convergent sum (convergent series) over the harmonics

$$u(x,t) = \sum_{n=0}^{\infty} a_n \hat{v}_n(x) e^{i\omega_n t}$$

where

$$a_n = \langle \hat{v}_n, u_0 \rangle_G, \qquad H\hat{v}_n = \omega_n \hat{v}_n$$

Non-self adjointness

(In)Completeness

In the **self-adjoint** (normal) case, any oscillation is a superposition of normal modes. In the **non-self adjoint** (non-normal) case, we don't have completeness for **generic** potentials[i].

Spectral decomposition

Spectral decomposition and excitation coefficients in the Schwarzschild case[ii]. Need for a systematic approach.

Keldysh expansion using the adjoint of L

Keldysh expansion from L and $L^\dagger$ introduced in previous work[iii].

$$L v_n = \omega_n v_n$$
$$L^\dagger w_n = \overline{\omega_n} w_n$$

[i] Warnick, (In)completeness of Quasinormal Modes, Acta Physica Polonica B

[ii] Ansorg, Macedo, Spectral decomposition of black-hole perturbations on hyperboloidal slices, Phys. Rev. D 93, 124016

[iii] Gasperin, Jaramillo, Energy scales and black hole pseudospectra: the structural role of the scalar product, Class. Quantum Grav. 39 115010

Introduction : quasinormal modes definitions and BH perturbation theory

Quasi-normal modes (QNMs)

Heuristics: Resonant response under linear perturbation characterized by complex frequencies. QNMs probe the background spacetime geometry
Lax-Phillips theory: QNMs as poles of the resolvent

Perturbation theory on Schwarzschild black hole

Scalar, electromagnetic and gravitational perturbations reduce to the following wave equation in the tortoise coordinates

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {r_*}^2} + V_\ell(r_*)\right)\phi_{\ell m} = 0 + \text{outgoing boundary conditions}$$

where V_ℓ depends on the type of perturbation (spin $s = 0, 1$ or 2).

Introduction : quasinormal modes definitions and BH perturbation theory

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Perturbation theory on Schwarzschild black hole

Scalar, electromagnetic and gravitational perturbations reduce to the following wave equation in the tortoise coordinates

$$\left(\frac{\partial^2}{\partial \bar{t}^2} - \frac{\partial^2}{\partial \bar{x}^2} + \lambda^2 V_\ell(\bar{x})\right)\phi_{\ell m} = 0 + \text{outgoing boundary conditions}$$

where V_ℓ depends on the type of perturbation (spin $s = 0, 1$ or 2).

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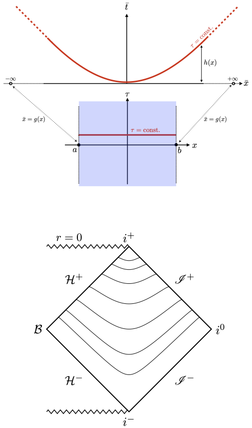
Compactified hyperboloidal approach



Illustrations of hyperboloidal slicings[iv]

Compactified hyperboloidal approach

$$\begin{cases} \bar{t} = t/\lambda \\ \bar{x} = r_*/\lambda \end{cases} \qquad \begin{cases} \bar{t} = \tau - h(x) \\ \bar{x} = g(x) \end{cases}$$

Killing vector

$$t^a = \partial_t = \frac{1}{\lambda}\partial_{\bar{t}} = \frac{1}{\lambda}\partial_\tau$$

QNMs as eigenvalues

QNMs as eigenvalues of a non-self adjoint operator

$$L v_n = \omega_n v_n$$

where

$$L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)$$

[iv] PhysRevX.11.031003, Jaramillo, Macedo, Al Sheikh and hyperboloid.al

Pöschl-Teller : a toy model (part 1)

Pöschl-Teller (1/2)

$$\left(\frac{\partial^2}{\partial \bar{t}^2} - \frac{\partial^2}{\partial \overline{x}^2} + V(\overline{x})\right)\phi = 0, \qquad V(\overline{x}) = V_0 \,\mathrm{sech}^2(\overline{x})$$

The following change of variable[v] defines a compactified hyperboloidal foliation :

$$\begin{cases} \tau = \bar{t} - \ln(\cosh \overline{x}) \\ x = \tanh^{-1}(\overline{x}) \end{cases}$$

$\bar{t}, \overline{x} \in \mathbb{R}; x \in [-1, 1]$

[v] Al Sheikh,Jaramillo,Macedo;2004.06434,Al Sheikh PhD,Bizoń,Chmaj,Mach;2002.01770

Pöschl-Teller : a toy model (part 2)

Pöschl-Teller (2/2)

First order reduction : $u(x,\tau) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ with $\psi := \partial_\tau \phi$,

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline \partial_x((1-x^2)\partial_x) - V_0 & -(2x\partial_x + 1) \end{array} \right) \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Differential equation :

$$\partial_\tau u = iLu, \qquad u(x, \tau = 0) = u_0(x)$$

Spectral problem : $Lv_n = \omega_n v_n$

Analytical Pöschl-Teller QNMs

$\phi_n(x) =$ Gegenbauer polynomials $C_n^{(i\omega_n + \frac{1}{2})}(x)$, $\qquad \omega_n = \pm \frac{\sqrt{3}}{2} + i\left(n + \frac{1}{2}\right)$

Pöschl-Teller quasi-normal frequencies

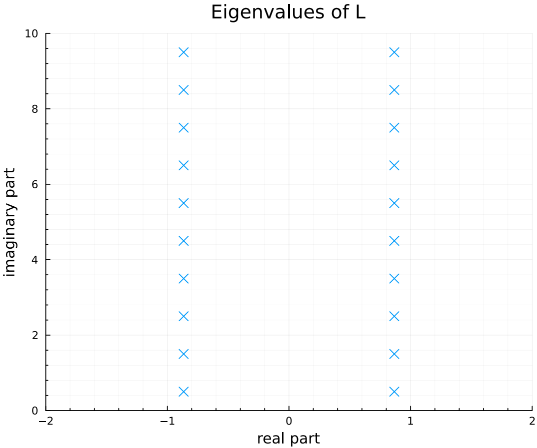


Figure: View of the Pöschl-Teller QNMs frequencies in the complex plane

Compactified hyperboloidal approach

Compactified hyperboloidal slicing

$$\begin{cases} \bar{t} = \tau - h(x) \\ \bar{x} = g(x) \end{cases} \qquad \begin{aligned} g\colon [a,b] &\to [-\infty, +\infty] \\ x &\mapsto g(x) = \bar{x} \end{aligned}$$

First order reduction

We define the field $\psi := \partial_\tau \phi$, the linear problem becomes

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array} \right) \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Outgoing boundary conditions

- Geometric interpretation : outgoing null cones
- Analytic interpretation : singular Sturm-Liouville operator, the boundary conditions are built-in as regularity conditions

Compactified hyperboloidal approach

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}_{iL} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

All the space derivatives are contained within the matrix.

$$L_1 = \frac{1}{w(x)} \left(\partial_x (p(x) \partial_x) - q(x)\right)$$

$$L_2 = \frac{1}{w(x)} \left(2\gamma(x)\partial_x + \partial_x \gamma(x)\right)$$

where

$$w(x) = \frac{g'(x)^2 - h'(x)^2}{|g'(x)|}$$

$$p(x) = \frac{1}{|g'(x)|}$$

$$q(x) = |g'(x)| V_\ell(x)$$

$$\gamma(x) = \frac{h'(x)}{|g'(x)|}$$

Scalar product and non-selfadjointness

The energy scalar product is related to the energy-momentum tensor of a complex scalar field on a Minkowski spacetime with a potential V_ℓ.

$$\left\langle \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix} \right\rangle_E = \frac{1}{2}\int_a^b w(x)\overline{\psi}_1\psi_2 + p(x)\partial_x\overline{\phi}_1\partial_x\phi_2 + q_\ell(x)\overline{\phi}_1\phi_2 dx$$

We use this to justify that L_2 is a dissipative term and is responsible for non-self adjointness

$$L^\dagger = L + \frac{1}{i}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 2\frac{\gamma(x)}{w(x)}\left(\delta(x-a)-\delta(x-b)\right)\end{array}\right)$$

Instability of the QNMs

The eigenvalues can be greatly perturbed upon a small perturbation of the potential

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Resolvent

$$\partial_\tau u(x,\tau) = iLu(x,\tau)$$

A Laplace transform yields,

$$(L-\omega)u(x,\omega) = iu_0(x)$$

Acting with the resolvent $R_L(\omega) = (L-\omega I)^{-1}$ on both sides we get

$$u(x,\omega) = i(L-\omega I)^{-1}u_0(x)$$

Keldysh's expansion of the resolvent

Consider the application

$$F\colon \Omega \to \mathcal{L}(\mathcal{H}, \mathcal{K})$$
$$\omega \mapsto F(\omega)$$

Assume $F(\omega)$ is a Fredholm operator. The transpose application of F is

$$F(\omega)^t\colon \mathcal{K}^* \to \mathcal{H}^*$$

The spectral problems are rewritten

$$F(\omega_n)v_n = 0, \qquad F(\omega_n)^t\alpha_n = 0, \qquad v_n \in \mathcal{H}, \alpha_n \in \mathcal{K}^*$$

Keldysh's theorem gives an expansion of the resolvent application[vi].

$$F^{-1}(\omega) = \sum_{\omega_n \in \Omega_0} \frac{\langle \widetilde{\alpha}_n, . \rangle}{\omega - \omega_n} v_n + H(\omega) \quad \text{with} \quad \left\langle \widetilde{\alpha}_n, \frac{dF}{d\omega}(\omega_n)(v_n) \right\rangle = 1.$$

[vi] Beyn, Latushkin, Rottmann-Matthes; 1210.3952

Keldysh's resonant expansion for non-generalized eigenvalue problems

We use the recipe with $F(\omega) = L - \omega I$, the spectral problems are :

$$(L - \omega_n I)v_n = 0, \qquad (L^t - \omega_n I)\alpha_n = 0, \qquad v_n \in \mathcal{H}, \alpha_n \in \mathcal{H}^*$$

The resolvent of L is constructed in a bounded domain Ω can be written

$$R_L(\omega) = (L - \omega I)^{-1} = \sum_{\omega_n \in \Omega_0} \frac{\langle \widetilde{\alpha}_n, . \rangle}{\omega - \omega_n} v_n + H(\omega)$$

On the other hand, the Laplace transform of the differential equation yields

$$(L - \omega)u(x, \omega) = i u_0(x)$$

The asymptotic resonant expansion is then found by multiplication by the resolvent and inverse Laplace transform

$$u(\tau, x) \sim \sum_n \langle \alpha_n, u_0 \rangle v_n(x) e^{i\omega_n \tau}, \qquad \text{with } \langle \alpha_n, v_n \rangle = 1$$

Asymptotic resonant expansion

Bound of the error of the Keldysh expansion

Given a bounded domain Ω in $\mathbb{C}$ and $R = \max_{\omega\in\Omega} \mathrm{Im}\{\omega\}$, we have

$$u(\tau, x) = \sum_{\mathrm{Im}\{\omega_n\}\leq R} \langle \alpha_n, u_0\rangle v_n(x) e^{i\omega_n \tau} + E_R(\tau; u_0)(x)$$

and

$$\|E_R(\tau; u_0)\|_E \leq \|u_0\|_E C_R(L) e^{-R\tau}$$

Notation

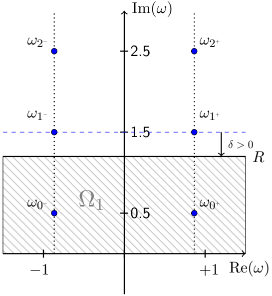
$$\mathcal{A}_n(x) = \underbrace{\langle \alpha_n, u_0\rangle}_{a_n} v_n(x), \qquad \text{with } \langle \alpha_n, v_n\rangle = 1$$

(see [vii] for)

$$\mathcal{A}_n^\infty = \mathcal{A}_n(x)|_{x=\text{null infinity}}$$

[vii] Ansorg and Macedo 10.1103/PhysRevD.93.124016

Asymptotic resonant expansion



Error for $1/2 < R < 3/2$

Summing over only the fundamental mode, we have the error

$$E_1(x,\tau) = u(x,\tau) - \sum_{n<\mathrm{Im}\{\omega_{1^\pm}\}} \mathcal{A}_n(x) e^{i\omega_n \tau}$$
$$= u(x,\tau) - \mathcal{A}_{0^-}(x) e^{i\omega_{0^-}\tau} - \mathcal{A}_{0^+}(x) e^{i\omega_{0^+}\tau}$$

Bound

$$\frac{\|E_1(\tau; u_0)\|_E}{\|u_0\|_E} \le C_1(L) e^{-\frac{3}{2}\tau}$$

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Cases of study

Pöschl-Teller (toy model)

$x \in [-1, 1]$

$$\begin{cases} h(x) = \log\bigl(1 - x^2\bigr) \\ g(x) = \operatorname{arctanh}(x) \end{cases}$$

Schwarzschild

$\sigma \in [0, 1]$, $\lambda = 4M$.

$$\begin{cases} h(\sigma) = \frac{1}{2}\left(\log\sigma + \log(1-\sigma) - \frac{1}{\sigma}\right) \\ g(\sigma) = \frac{1}{2}\left(\frac{1}{\sigma} + \log(1-\sigma) - \ln\sigma\right) \end{cases}$$

$$L_1 = \frac{1}{2(1+\sigma)}\left[\partial_\sigma\left(2\sigma^2(1-\sigma)\partial_\sigma\right) - 2\ell(\ell+1) - (1-s^2)\sigma\right]$$

$$L_2 = \frac{1}{2(1+\sigma)}[2(1-2\sigma^2)\partial_\sigma - 4\sigma]$$

Cases of study

Cases	$f(r)$	potential for $s=0$, 1 or 2 odd (axial perturbations)	potential for $s=2$ even (polar perturbations)
Pöschl-Teller (toy model)	$V_0 \operatorname{sech}^2\left(\frac{r}{b}\right)$		
Schwarzschild	$1-\frac{2M}{r}$	$f(r)\left(\frac{\ell(\ell+1)}{r^2}+(1-s^2)\frac{2M}{r^3}\right)$	$f(r)\frac{2}{r^3}\frac{9M^3+3c^2Mr^2+c^2(1+c)r^3+9M^2cr}{(3M+cr)^2}$
Schwarzschild-(Anti) de Sitter	$1-\frac{2M}{r}-\frac{\Lambda r^2}{3}$	$f(r)\left[\frac{\ell(\ell+1)}{r^2}+(1-s^2)\left(\frac{2M}{r^3}-\frac{2-s}{3}\Lambda\right)\right]$	$\frac{2f(r)}{r^3}\frac{9M^3+3c^2Mr^2+c^2(1+c)r^3+3M^2(3cr-\Lambda r^3)}{(3M+cr)^2}$

Table 1 . Expressions for the potential in the three cases we cover. In these expressions, the cosmological constant may be positive or negative. We denote $c=\frac{(\ell-1)(\ell+2)}{2}$.

Schwarzschild-de Sitter

p vanish linearly at the event horizon and at the cosmological horizon, its expression depends on the surface gravity[viii]

Schwarzschild-Anti de Sitter

Reflexive (Dirichlet) boundary conditions imposed at the AdS boundary : acts like a box that confines the field. There are only dissipations at the event horizon[ix]

[viii] Sarkar,Rahman and Chakraborty 10.1103/PhysRevD.108.104002; Destounis, Boyanov and Macedo 10.1103/PhysRevD.109.044023

[ix] Areán, Fariña and Landsteiner 10.1007/JHEP12(2023)187; Boyanov,Cardoso,Destounis and Jaramillo 10.1103/PhysRevD.109.064068

Numerical methods

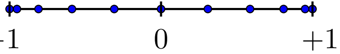


Figure: Chebyshev-Lobatto grid, $x_j = \cos\left(\frac{\pi j}{N}\right)$

The discretized counterpart of ϕ is a vector with $N+1$ entries:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \\ \phi_{N+1} \end{bmatrix}, \qquad \phi(x_j) = \phi_j$$

Numerical instability: we work with arbitrary precision numerics, typically 10^{-800}

$L : (2N+2) \times (2N+2)$ entries (matrix)

$u : (2N+2)$ entries (column vector)

Time evolutions

Sector	Parameter
Chebyshev-Lobatto grid size	N
arbitrary decimal precision	precision
ODE/DAE solver (numerical time evolution)	dt increment
	tolerance
	algorithm

Note. — The decimal precision controls the arithmetic precision of real or complex floating numbers. The solver's dt is fixed to 10^{-7}. The precision of the ODE solver is controlled by a parameter named ”tolerance”. Furthermore, the solver requires an algorithm that corresponds to the discretization scheme of the time derivative. we chose to exploit Julia's automatic stiffness detection feature and we figured out the best choice in terms of accuracy and execution time is probably AutoVern9(Rodas5P()).

Figure: Parameters involved in the simulations

Time evolutions

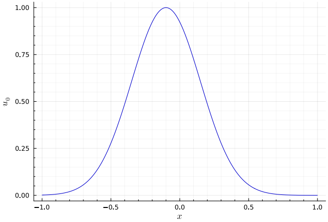
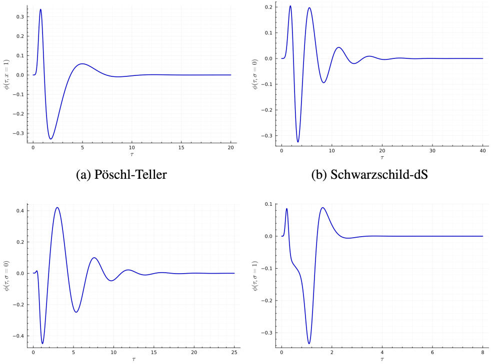


Figure: Initial data



(a) Pöschl-Teller

(b) Schwarzschild-dS

(c) Schwarzschild

(d) Schwarzschild-AdS

Figure: Waveforms at future null infinity (event horizon for the AdS case)

Spectra

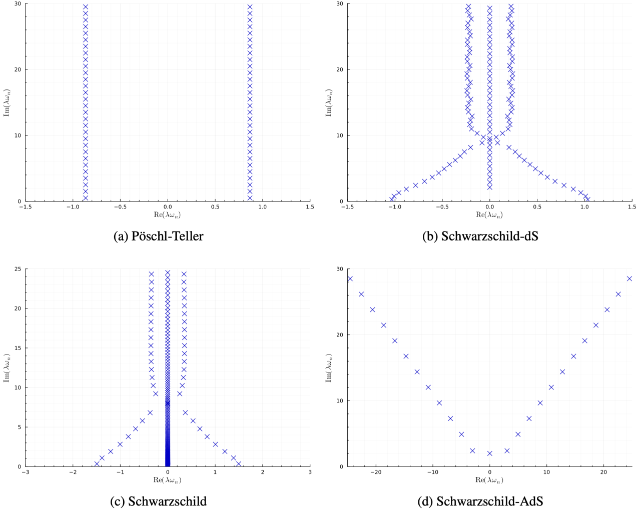
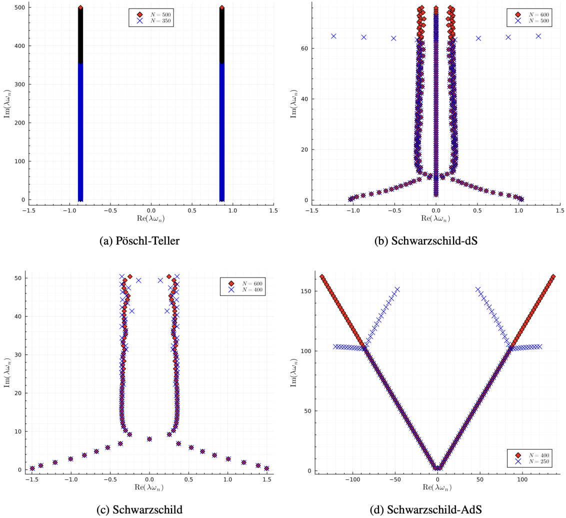


Figure: Spectra of the cases of study (event horizon for the AdS case)

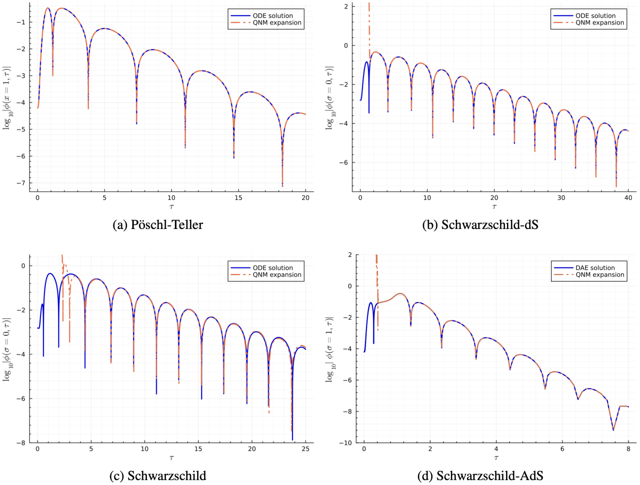
Spectra



(a) Pöschl-Teller (b) Schwarzschild-dS (c) Schwarzschild (d) Schwarzschild-AdS

Figure: Spectra of the cases of study for different gridsizes N.

Comparing the time and the spectral domain analysis

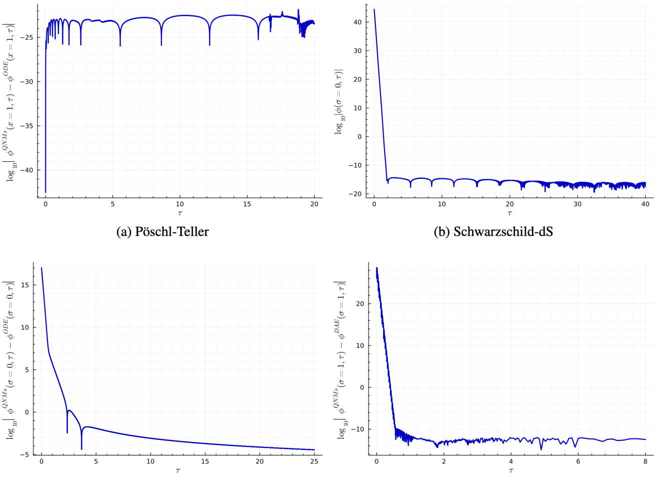


(a) Pöschl-Teller (b) Schwarzschild-dS

(c) Schwarzschild (d) Schwarzschild-AdS

Figure: We compare the ODE solution and the Keldysh QNM expansion at future null infinity (event horizon for the AdS case).

Comparing the time and the spectral domain analysis

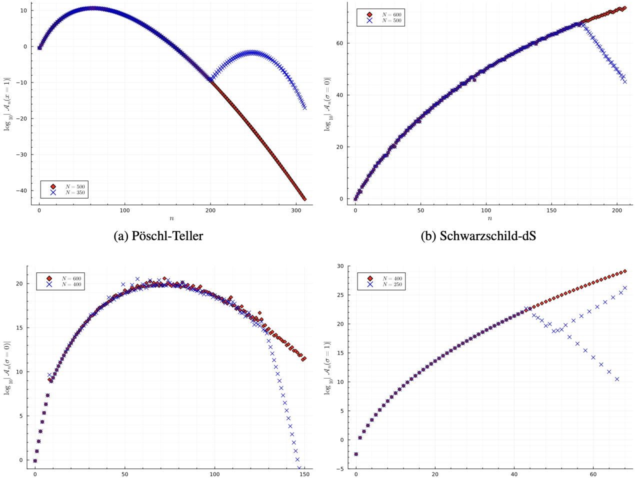


(a) Pöschl-Teller (b) Schwarzschild-dS

(c) Schwarzschild (d) Schwarzschild-AdS

Figure: Difference between the ODE solution and the Keldysh QNM expansion at future null infinity.

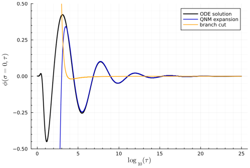
Coefficients of the timeseries at future null infinity



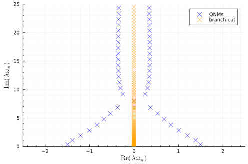
(a) Pöschl-Teller (b) Schwarzschild-dS (c) Schwarzschild (d) Schwarzschild-AdS

Figure: Log plot of the modulus of the coefficients $\mathcal{A}_n^\infty$.

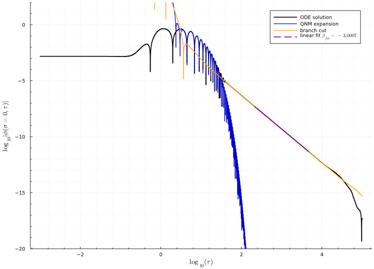
Schwarzschild case : separating tails and QNMs



(a) Waveform at future null infinity $\phi(\tau,\sigma)|_{\sigma=0}$



(b) Eigenvalue plot



(c) Waveform $\phi(\tau,\sigma)|_{\sigma=0}$ on a log-log plot

Figure: Separating branch cut and QNMs in the Schwarzschild case.

Polynomial tails and branch cut

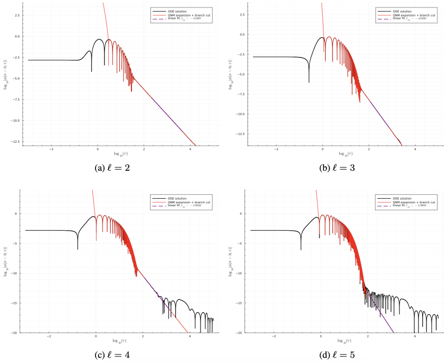


Figure: Polynomial tails in the Schwarzschild case.

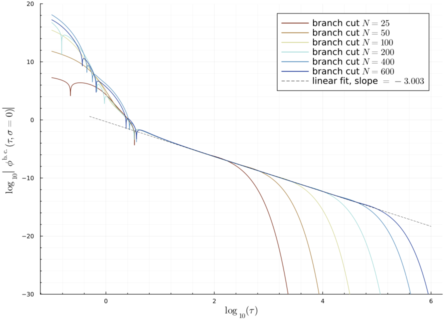


Figure: The bigger N, the longer the tail. ($\ell = 2$ in this figure).

Dynamics from the (exponentiated) evolution operator

Finite rank case (matrix case)

We assume the matrix $\boldsymbol{L}$ can be diagonalized: $\boldsymbol{L} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where

$$\begin{aligned}
\boldsymbol{D} &= \text{diag}(\omega_1, \omega_2, ..., \omega_{2N+1}, \omega_{2N+2}) \\
\boldsymbol{P} &= \left(\ \boldsymbol{v}_1 \ \middle| \ \boldsymbol{v}_2 \ \middle| \ ... \ \middle| \ \boldsymbol{v}_{2N+1} \ \middle| \ \boldsymbol{v}_{2N+2} \ \right) \\
\left(\boldsymbol{P}^{-1}\right)^t &= \left(\ \boldsymbol{\alpha}_1 \ \middle| \ \boldsymbol{\alpha}_2 \ \middle| \ ... \ \middle| \ \boldsymbol{\alpha}_{2N+1} \ \middle| \ \boldsymbol{\alpha}_{2N+2} \ \right)
\end{aligned}$$

- $\boldsymbol{v}_n$ are eigenvectors of $\boldsymbol{L}$
- $\boldsymbol{\alpha}_n$ are eigenvectors of $\boldsymbol{L}^t$ such that $\langle \boldsymbol{\alpha}_n, \boldsymbol{v}_n \rangle = 1$

The "formal" solution of $\partial_\tau \boldsymbol{u} = i\boldsymbol{L}\boldsymbol{u}$ is $e^{i\boldsymbol{L}\tau}\boldsymbol{u}_0$, it corresponds to the sum over **all** the eigenvalues,

$$\boldsymbol{P}e^{i\boldsymbol{D}\tau}\boldsymbol{P}^{-1}\boldsymbol{u_0} = \sum_{n=1}^{2N+2} \frac{\langle \boldsymbol{\alpha}_n, \boldsymbol{u_0} \rangle}{\langle \boldsymbol{\alpha}_n, \boldsymbol{v_n} \rangle} e^{i\omega_n \tau} \boldsymbol{v_n}$$

Role of overtones: Pöschl-Teller

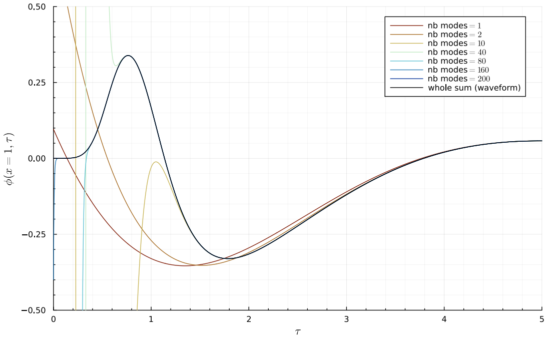


Figure: We show we recover the early times of the waveform by adding enough overtones. The panel on the right is a zoom.

Convergence of the series ? We can describe the waveform using 310 modes with a maximum error $\approx 10^{-40}$, this begs the question whether the series is convergent or not. What meaning do we give to the word "convergent" here ?

Role of overtones : Pöschl-Teller

$$u(\tau,x) = \sum_{\operatorname{Im}\omega_n \leq R} \mathcal{A}_n(x) e^{i\omega_n \tau} + E_R(\tau; u_0)(x), \qquad \|E_R(\tau;u_0)\|_E \leq \|u_0\|_E C_R(L) e^{-R\tau}$$

We want to assess the convergence of the sum as a series. does

$$\forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n > M, \left\|E_n(\tau;u_0)\right\|_E < \varepsilon ?$$

$\longrightarrow$We plot $\left\|E_n(\tau;u_0)\right\|_E$ as a function of n (the number of QNMs in the truncated sum)

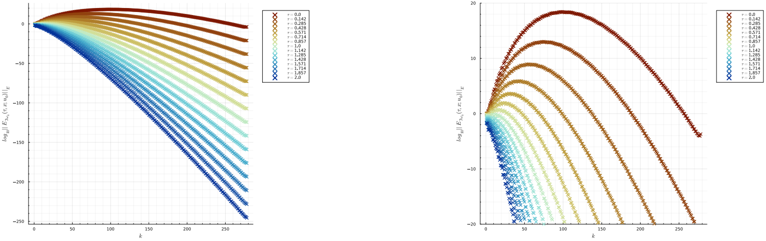


Figure: Norm of the error as we add more terms to the QNM expansion. A color corresponds to a time τ.

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Pseudospectrum

Given a perturbation δL of L of norm ε, what is the set of complex numbers λ which are actual eigenvalues of some perturbed operator $L + \delta L$?

Perturbative approach

$$\sigma^{\varepsilon}(L) = \{\lambda \in \mathbb{C}, \exists \delta L \in M_n(\mathbb{C}), \|\delta L\| < \varepsilon : \lambda \in \sigma(L + \delta L)\}$$

Resolvent norm approach

$$\sigma^{\varepsilon}(L) = \{\lambda \in \mathbb{C} : \|R_L(\lambda)\| = \left\|(\lambda I - L)^{-1}\right\| > 1/\varepsilon\}$$

Pseudospectrum in the self-adjoint case (Pöschl-Teller with $L_2 = 0$)

The colors correspond to $\log_{10} \varepsilon$.
The contour lines form circles centered on the eigenvalues and horizontal lines far away from the eigenvalues.

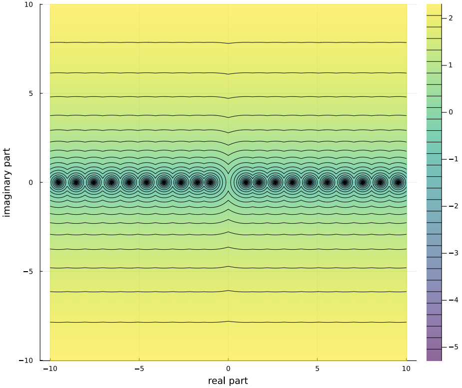

Figure: Pseudospectrum in the self adjoint case.

Pseudospectrum (Pöschl-Teller)

The contour lines are open and the eigenvalue can migrate very far from the eigenvalues of the non perturbed operator.
Issue : The (numerical) energy pseudospectrum doesn't converge with N

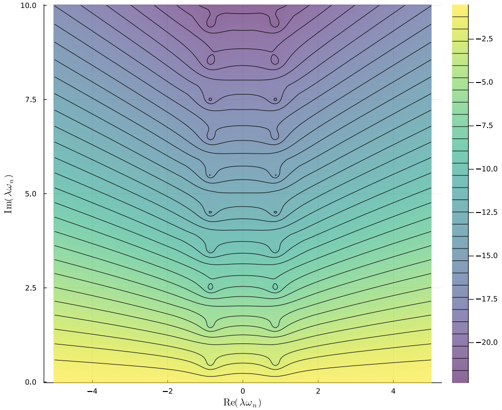

Figure: Pöschl-Teller energy pseudospectrum.

H^p-pseudospectrum

H^p-QNMs

H^p-QNMs are eigenfunctions of the H^p-regular operator

$$L_p \colon H^p \times H^{p-1} \to H^p \times H^{p-1}$$
$$(\phi, \psi) \mapsto L(\phi, \psi)$$

they constitute a finite set below $\mathrm{Im}(\lambda) < a + \kappa\left(p - \frac{1}{2}\right)$ with κ the surface gravity and some constant a. QNMs contained in the first p bands of width κ are required to have H^p regularity. We introduce a norm that make the Pöschl-Teller pseudospectrum converge in bands and increases the regularity of the QNMs in these bands.[x]

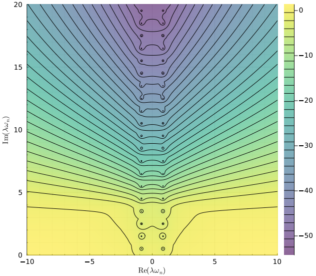

Figure: H^4-pseudospectrum.

$$\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{H^p}^2 := \sum_{j=0}^{p} \left\| \begin{pmatrix} \partial_x^j \phi \\ \partial_x^j \psi \end{pmatrix} \right\|_E^2$$

[x] Warnick;1306.5760, Boyanov,Cardoso,Destounis,Jaramillo;2312.11998

H^8-pseudospectra (Pöschl-Teller case)

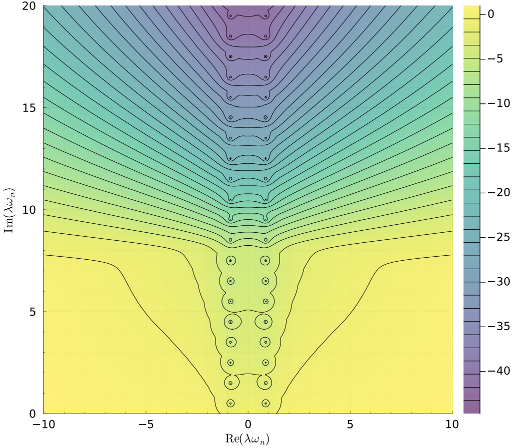

Figure: H^8-pseudospectrum.

Convergence test of the H^p-pseudospectrum

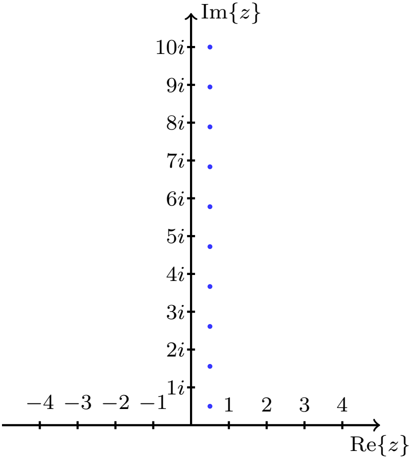

Figure: We pick some points in the complex plane

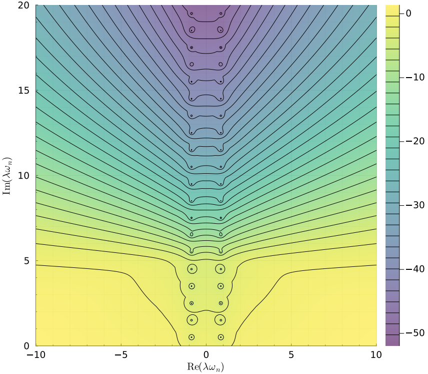

Figure: H^5-pseudospectrum

Convergence of the H^1-pseudospectrum

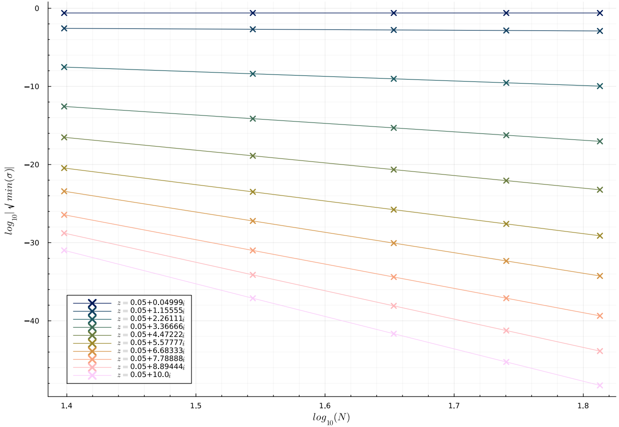

Figure: Norm of the resolvent for the H^1 norm and different z in the complex plane.

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An approach to the convergence of the asymptotic series (motivation)

Error bound (theorem)

$$\frac{\|E_n(\tau; u_0)\|_E}{\|u_0\|_E} \leq C_n e^{-(n+1/2)\tau}$$

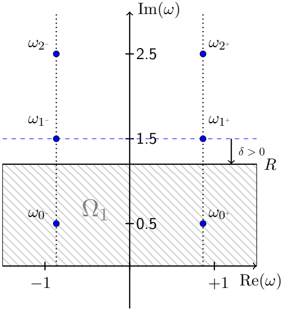
n is the number of modes in the truncated expansion
C_n doesn't depend on u_0

Norm of a matrix A with respect to the scalar product $\langle ., . \rangle_G$

$$\|A\|_G = \max_{u_0} \frac{\|Au_0\|_G}{\|u_0\|_G}$$

Can we estimate C_n without fixing any particular u_0 ?
Idea: Get rid of u_0 by transforming E_n into a matrix and then compute its norm.

An approach to the convergence of the asymptotic series



$$
\begin{aligned}
\boldsymbol{E}_1(\tau)\boldsymbol{u}_0 &= \boldsymbol{u}(\tau) - \sum_{\omega_n \in \Omega_1} \mathcal{A}_n e^{i\omega_n \tau} \\
&= \boldsymbol{P}\,\mathrm{diag}\left(e^{i\omega_{0^-}\tau}, e^{i\omega_{0^+}\tau}, e^{i\omega_{1^-}\tau}, e^{i\omega_{1^+}\tau}, ..., \right)\boldsymbol{P}^{-1}\boldsymbol{u}_0 \\
&\qquad\qquad - \boldsymbol{P}\,\mathrm{diag}\left(e^{i\omega_{0^-}\tau}, e^{i\omega_{0^+}\tau}, 0, 0, ..., \right)\boldsymbol{P}^{-1}\boldsymbol{u}_0 \\
&= \boldsymbol{P}\,\mathrm{diag}\left(0, 0, e^{i\omega_{1^-}\tau}, e^{i\omega_{1^+}\tau}, ..., \right)\boldsymbol{P}^{-1}\boldsymbol{u}_0
\end{aligned}
$$

An approach to the convergence of the asymptotic series

We compute the norm of the matrix

$$\boldsymbol{E}_1(\tau) = \boldsymbol{P}\,\mathrm{diag}\left(0, 0, e^{i\omega_{1^-}\tau}, e^{i\omega_{1^+}\tau}, ...,\right)\boldsymbol{P}^{-1}$$

We traded a depency on u_0 for a depency on τ. We chose $\tau = 5$ for the next two figures.
The norm $\|\boldsymbol{E}_n(\tau)\|_{H^p}$ converges as the gridsize N increases <u>if $n \leq p$</u>.

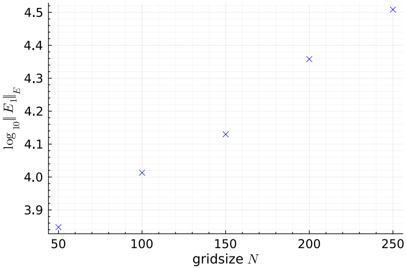


Figure: $\log_{10}$ of the energy norm of the error $E_1(\tau = 5)$

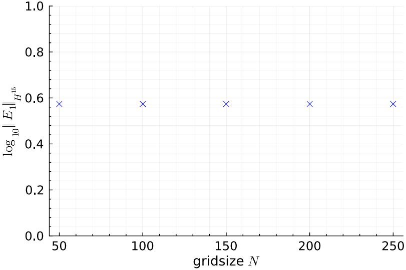


Figure: $\log_{10}$ of the H^{15}-norm of the error $E_1(\tau = 5)$

An approach to the convergence of the asymptotic series

Define

$$\boldsymbol{E}_n(\tau) = e^{-(n+1/2)\tau}\,\widetilde{\boldsymbol{E}_n}(\tau)$$

and

$$C_n = \max_{\tau\leq\tau_c}\left\|\widetilde{\boldsymbol{E}_n}(\tau)\right\|_{H^p}$$ (note: this is called C_n^∞ in the appendix)

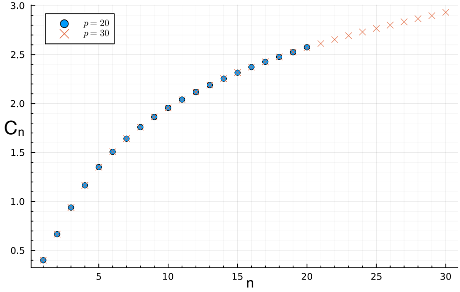


Figure: $\log_{10}$ of C_n for different H^p norms

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Conclusions

- Unique expansion at null infinity
- Keldysh's approach generalizes previous QNM expansion schemes (higher dimensions, tails, ...)
- Agnostic nature of the Keldysh QNM expansion : the expansion is independant of a scalar product ("dual pairing" $\langle ., . \rangle$ notion instead)
- Polynomial tails are recovered and follow the Price law
- Role of overtones at early times of the waveforms
- H^p-pseudospectra converge according to Warnick's criterion
- An insight into the role of regularity for the convergence of the asymptotic expansion
- Dynamics from the evolution operator amounts to Keldysh over all the eigenvalues of the matrix

Conclusions

Thanks for your attention !