A fairy-TAIL story

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[2406.17018]

strong. niels bohr institute
KØBENHAVNS UNIVERSITET
Danmarks Grundforskningsfond
Danish National Research Foundation
erc
European Research Council
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Settings


m_1
m_2
Inspiral
Merger
Ringdown
(~0.2 ms if M=50 solar masses)
Time
Final state:
• Equilibrium configuration
$Strain\,(10^{-21})$
$time$ [M]
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−700
−600
−500
−400
−300
−200
−100
0
100

Question...

Final state:

- **Equilibrium configuration**

What is the late-time signal ≡ tail?

Summary

- Brief introduction on tails in perturbation theory
- Tails phenomenology and model in extreme mass-ratio mergers
- Extracting tails in fully non-linear numerical relativity

Why it matters?

Foundational problem in GR

- Classical soft graviton theorems predicting new tails [Sen, 2408.08851]
- Tails related to peeling properties (/ lack of) [Gajic and Kehrberger, 2202.04093]

$\boxed{\text{Probe of spacetime asymptotic structure}}$

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Probe of spacetime asymptotic structure

Astrophysical implications

- Enhancement with eccentricity makes the tail **potentially observable**
  - Plenty possible channels of highly eccentric mergers
  - Could give constraints on eccentric inspiral parameters

Relevant for binary formation and population studies

Initial-data driven tails

First order
perturbation theory

$$\left[\partial_t^2 - \partial_{r_*}^2 + V_{\ell m}(r_*)\right] \Psi_{\ell m}(t, r_*) = 0$$

$$\Psi_{\ell m}(t=0,r) = \psi_0 \qquad \partial_t \Psi_{\ell m}(t=0,r) = \zeta_0$$

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Solution

$$\int dt' \int dr'_* \underbrace{G_{\ell m}(t - t'; r_*, r'_*)}_{\text{Tail propagator}} \; \underbrace{Q_{\ell m}(t', r'_*)}_{\text{Initial-data source}}$$

Initial-data driven tails

- find tail propagator, in (complex) ω domain

$$\left[\partial_{r_*}^2 + \omega^2 - V_{\ell m}(r_*)\right] \tilde{G}_{\ell m}(\omega; r_*', r_*) = \delta(r_* - r_*')$$

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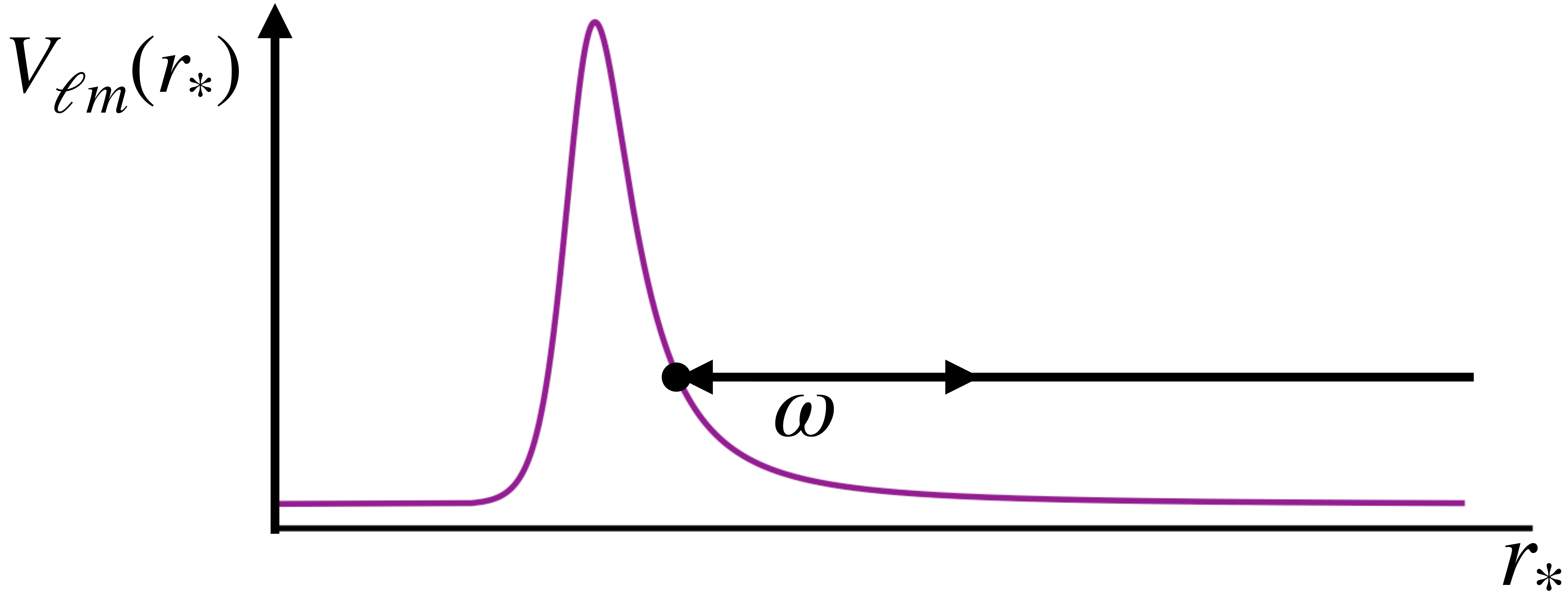
Originated from backscattering of small ω signal from long-range potential



[Price, Phys. Rev. D 5, 2419] , [Leaver, Phys. Rev. D 34, 384]

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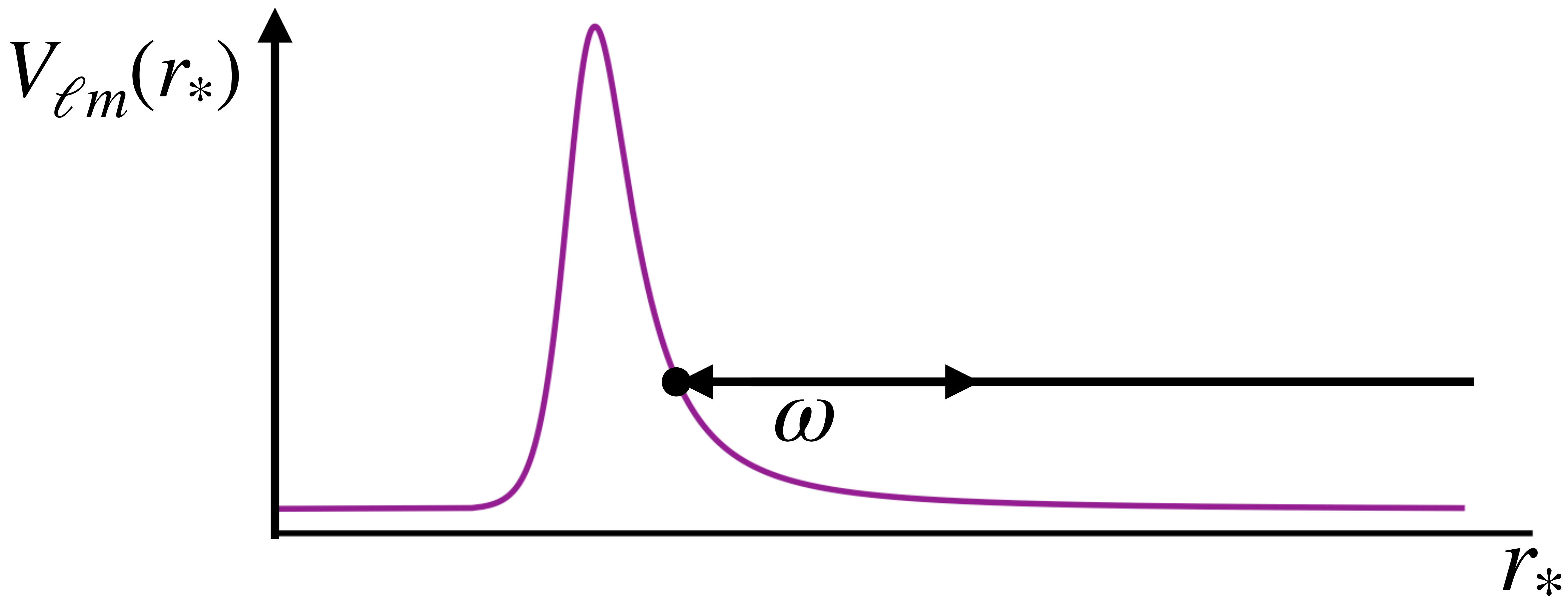

Expand in large r and small ω

[Price, Phys. Rev. D 5, 2419] , [Leaver, Phys. Rev. D 34, 384]

Initial-data driven tails

Expansion in large r...

$$\left[\partial_r^2 + \omega^2\left(1 - \frac{4M}{r^2}\right) - \frac{\ell(\ell+1)}{r^2}\right]\tilde{G}_{\ell m}(\omega; r', r) = \delta(r - r')$$

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$$G_\ell(t - t'; r, r') \propto \int_\Gamma d\omega\ e^{-i\omega(t-r-t')}\ (\omega r')^{\ell+1}\ \left[\ \log(2\omega)\ \ + \ldots\right]$$

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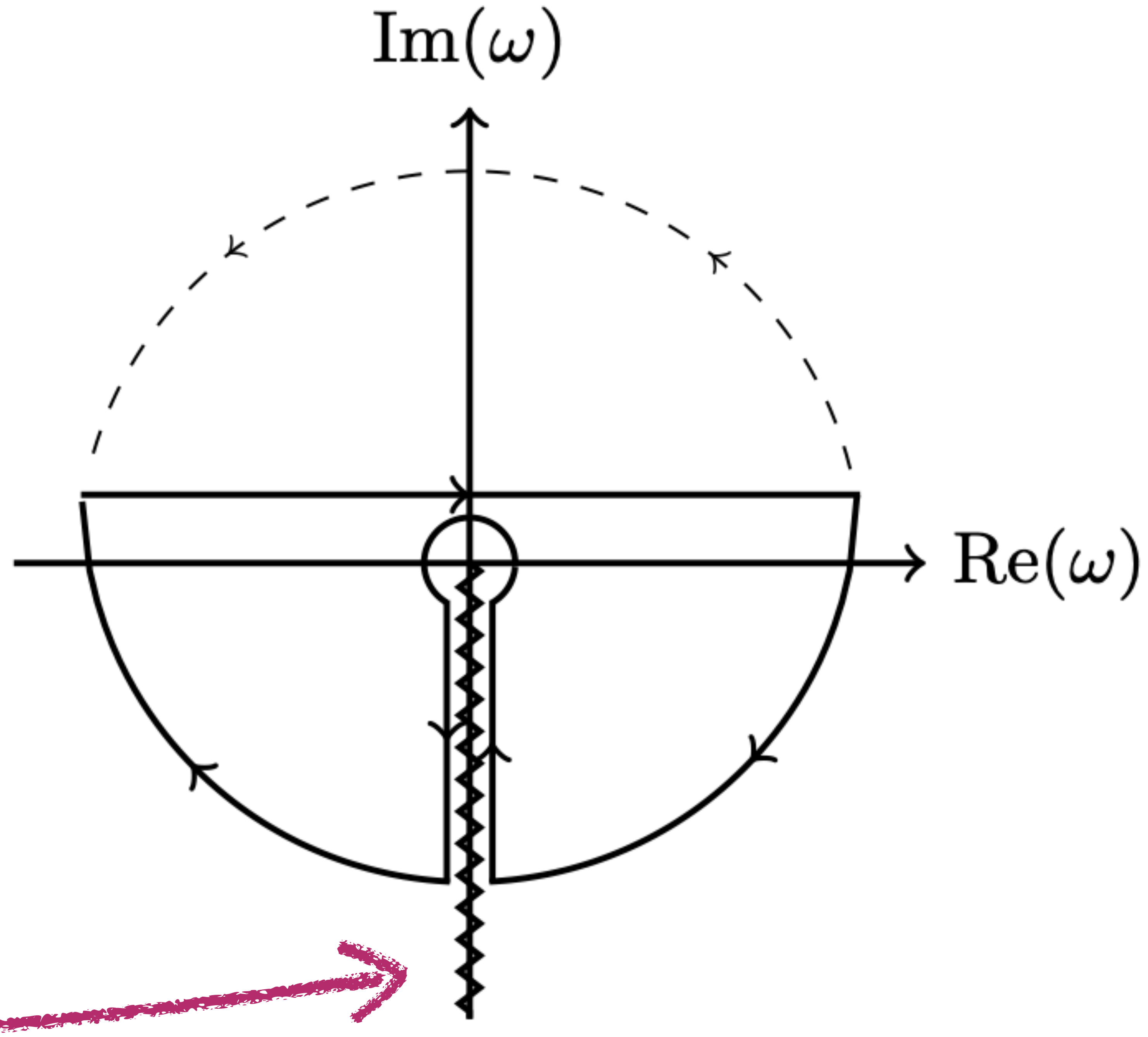

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Initial-data driven tails

Prediction at $\mathscr{I}^+$:

- $\boxed{\Psi_{\ell m} = \dfrac{A_{\text{tail}}}{\tau^{\ell+2}}}$

$$\tau \equiv t - r_*$$

$$r_* \equiv r + 2M\ln(r - 2M)$$

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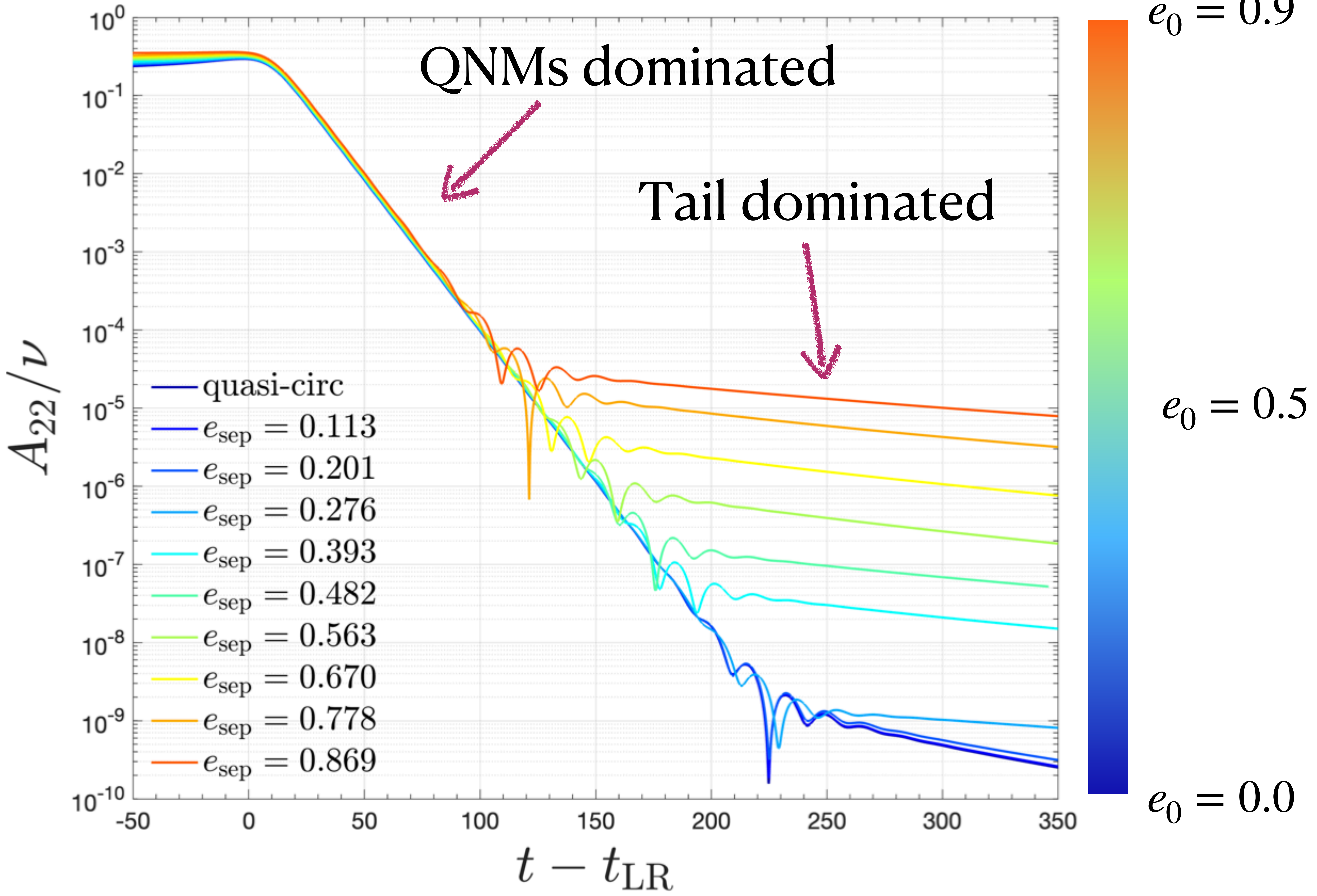
1) Suppressed
2) Not clear what is the information content

[Price, Phys. Rev. D 5, 2419] , [Leaver, Phys. Rev. D 34, 384]

An exciting phenomenology

[Albanesi et al, Phys. Rev. D 108, 084037]



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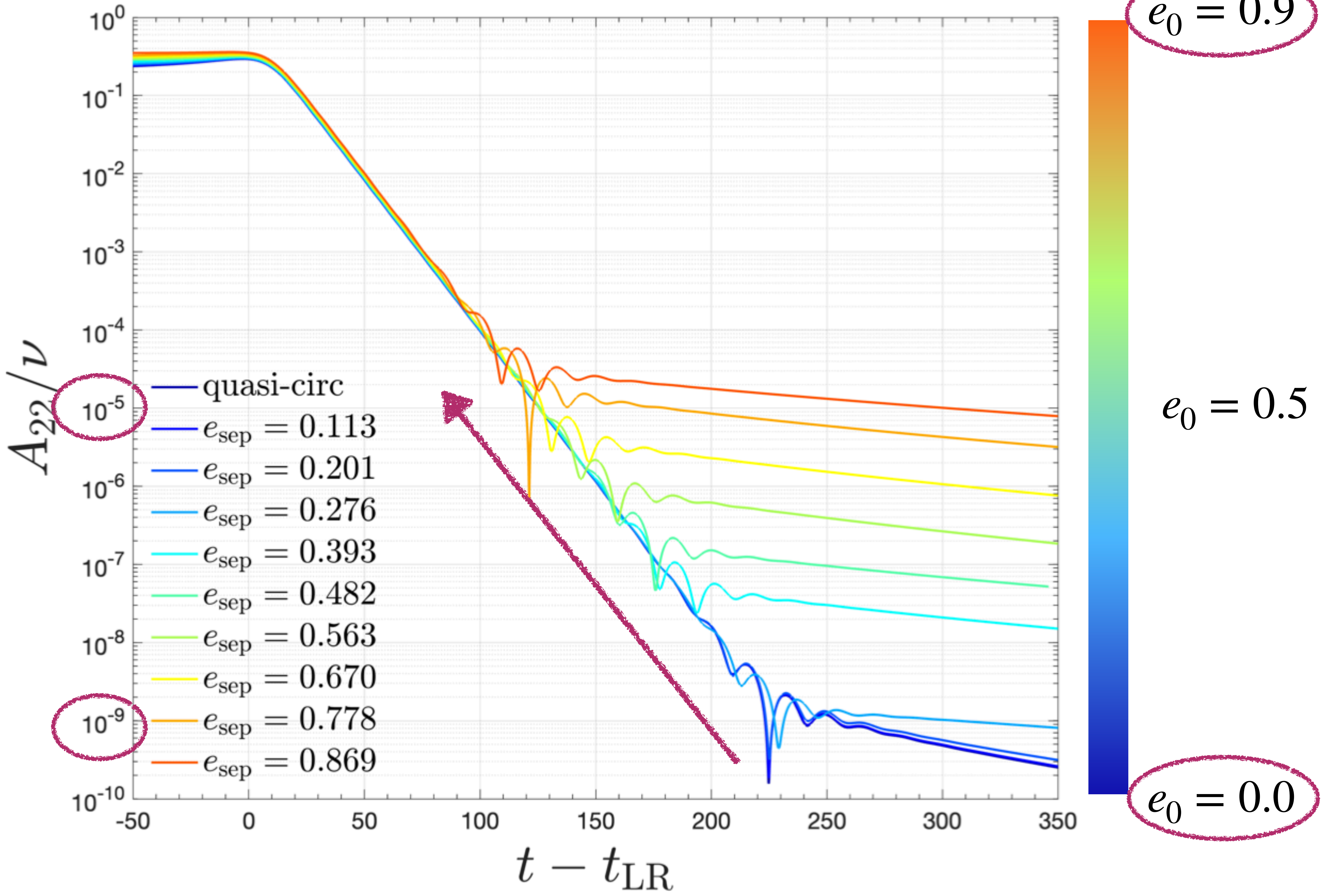

Amplitude

enhanced of several orders of magnitude by eccentricity

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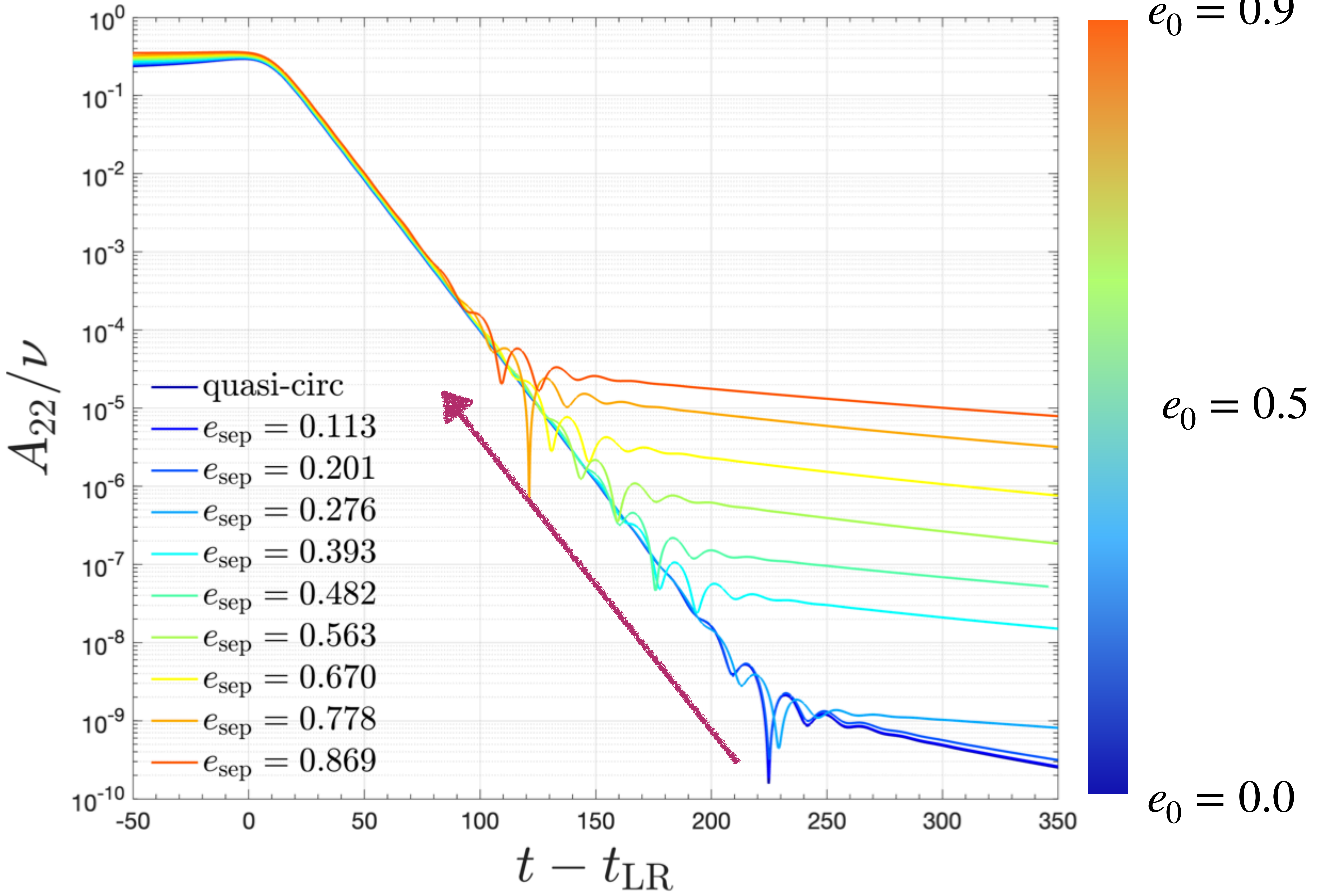

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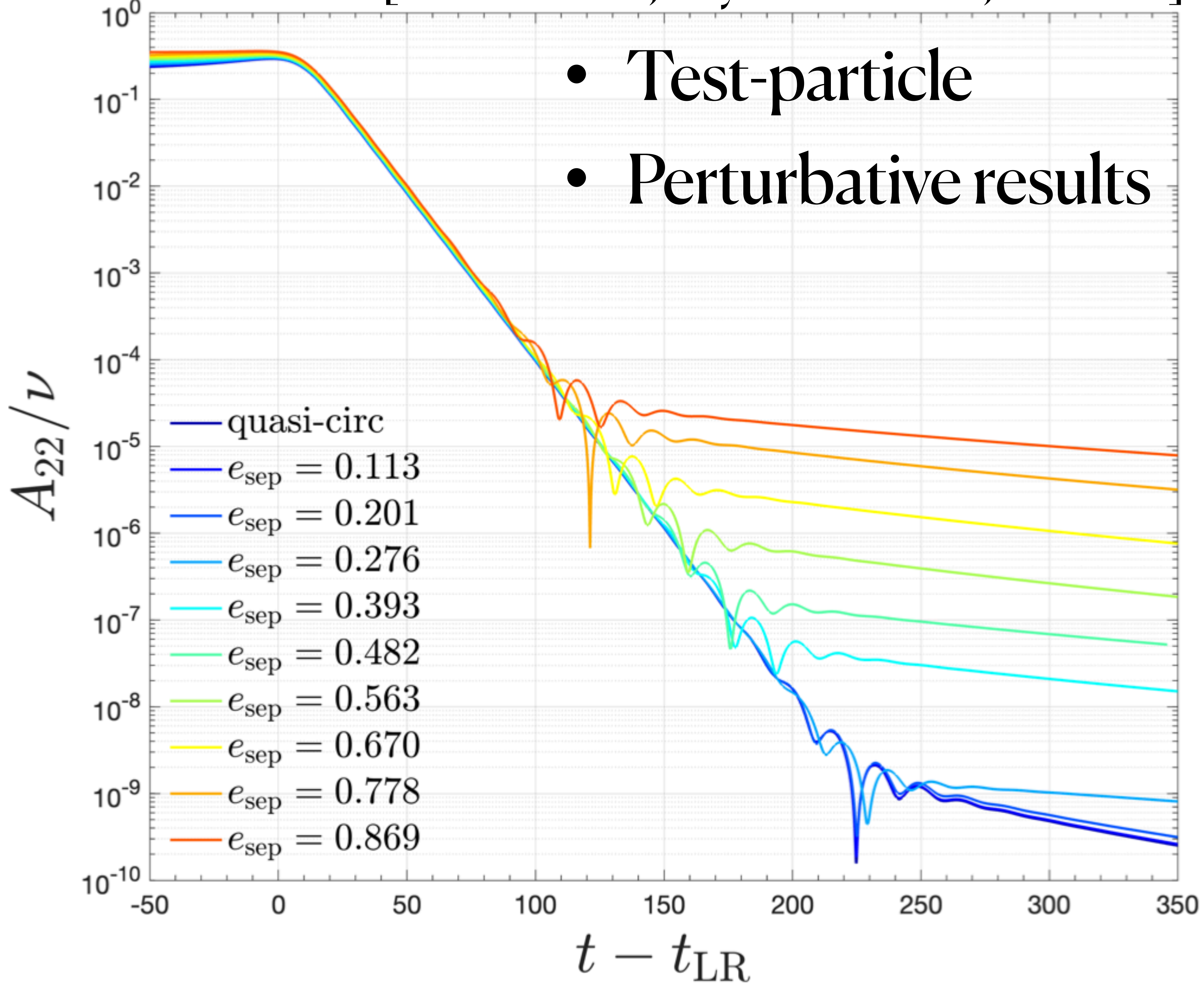
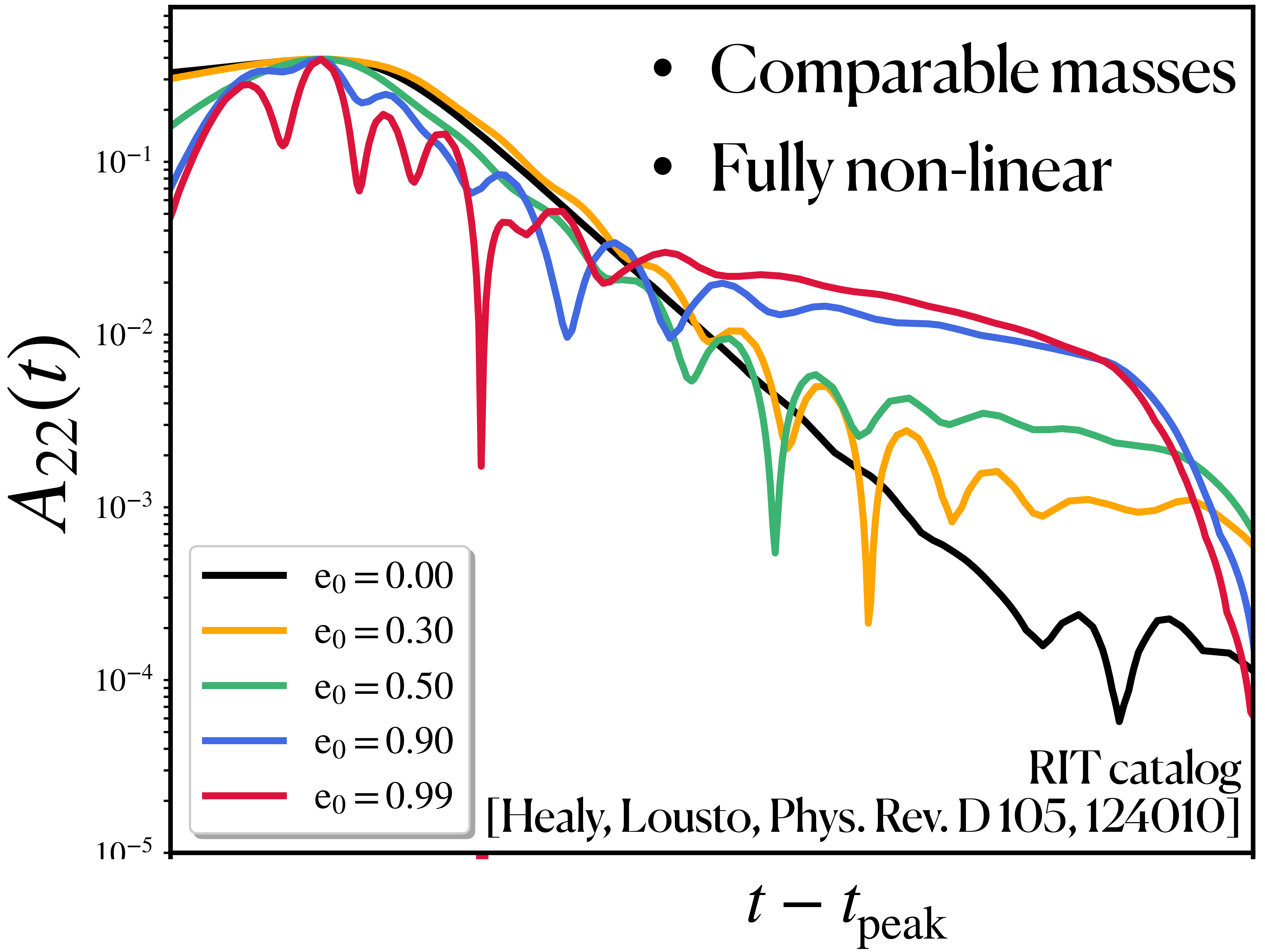
What happens for comparable masses

An exciting journey: EMR vs comparable masses

[Albanesi et al, Phys. Rev. D 108, 084037]



[Carullo and **De Amicis**, 2310.12968]



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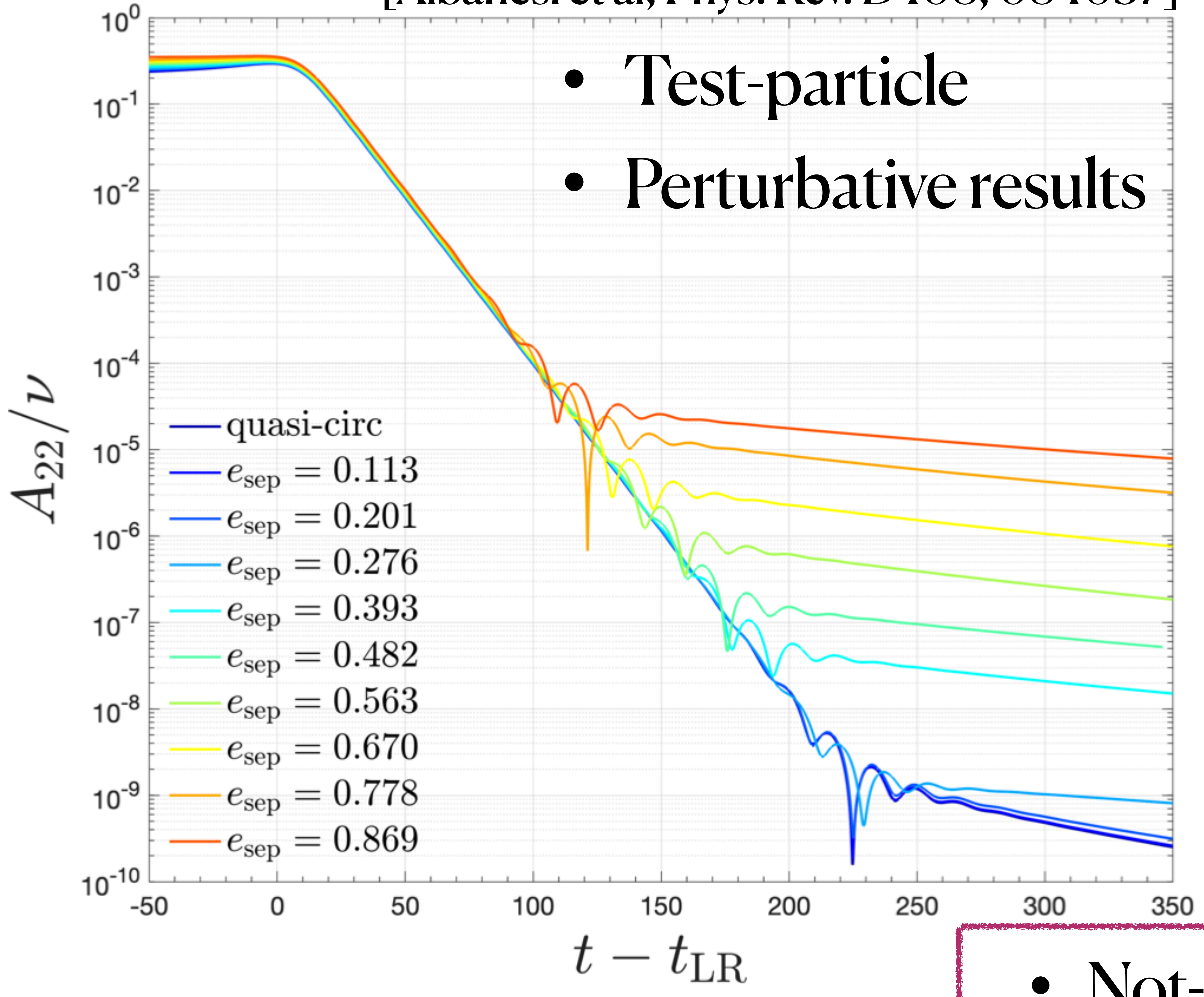

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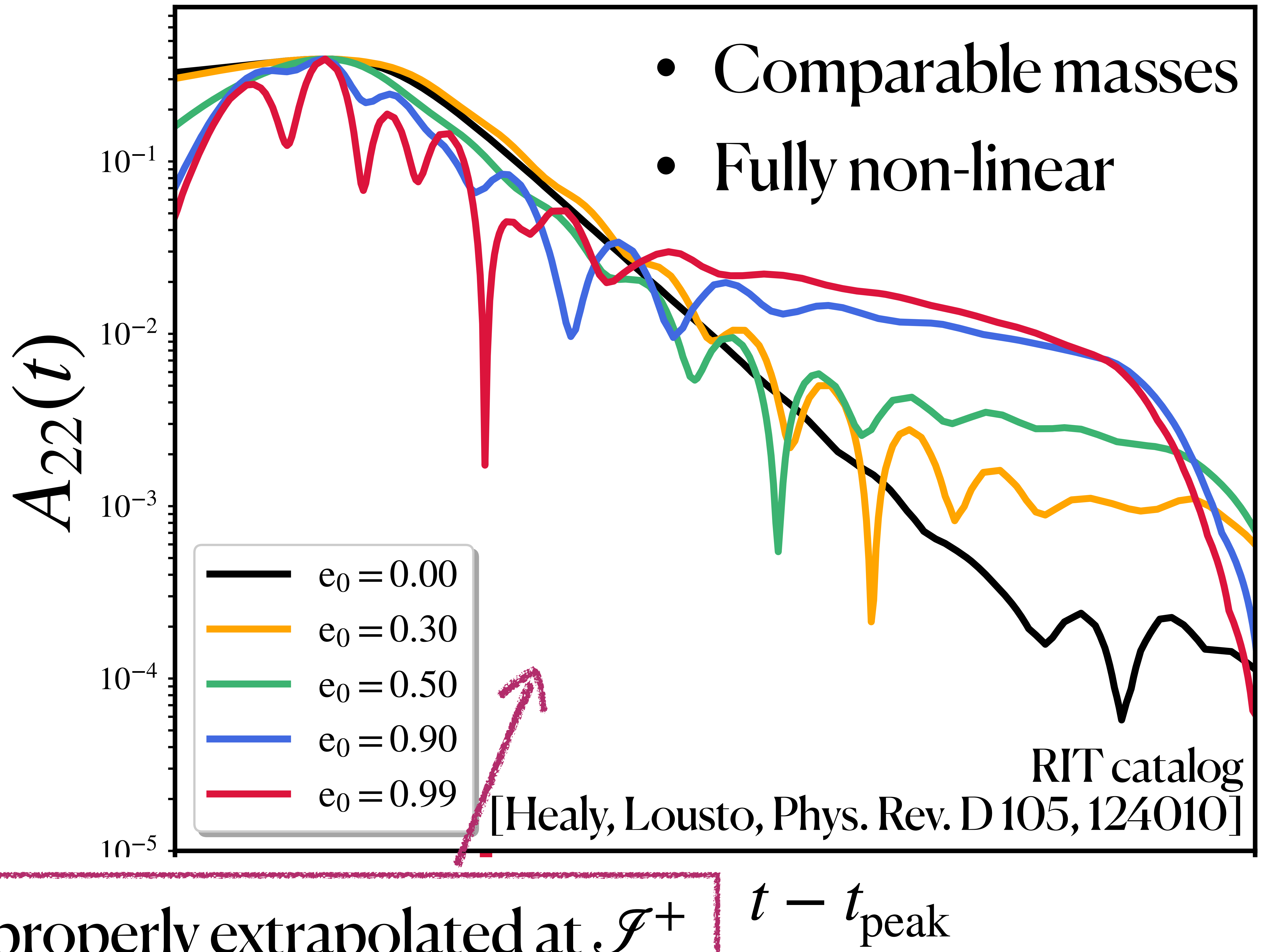
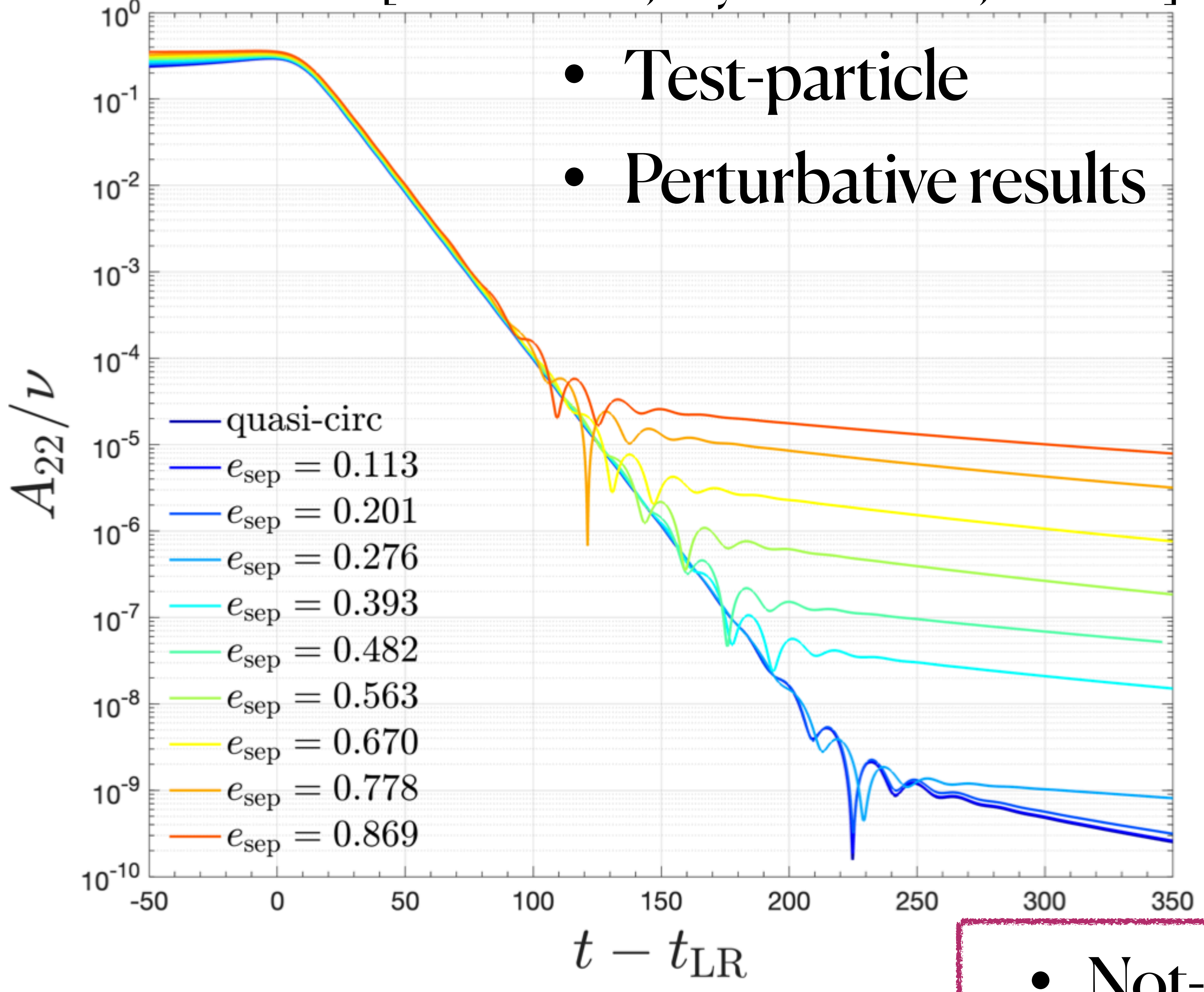


- Not-properly extrapolated at $\mathscr{I}^+$

An exciting journey: EMR vs comparable masses

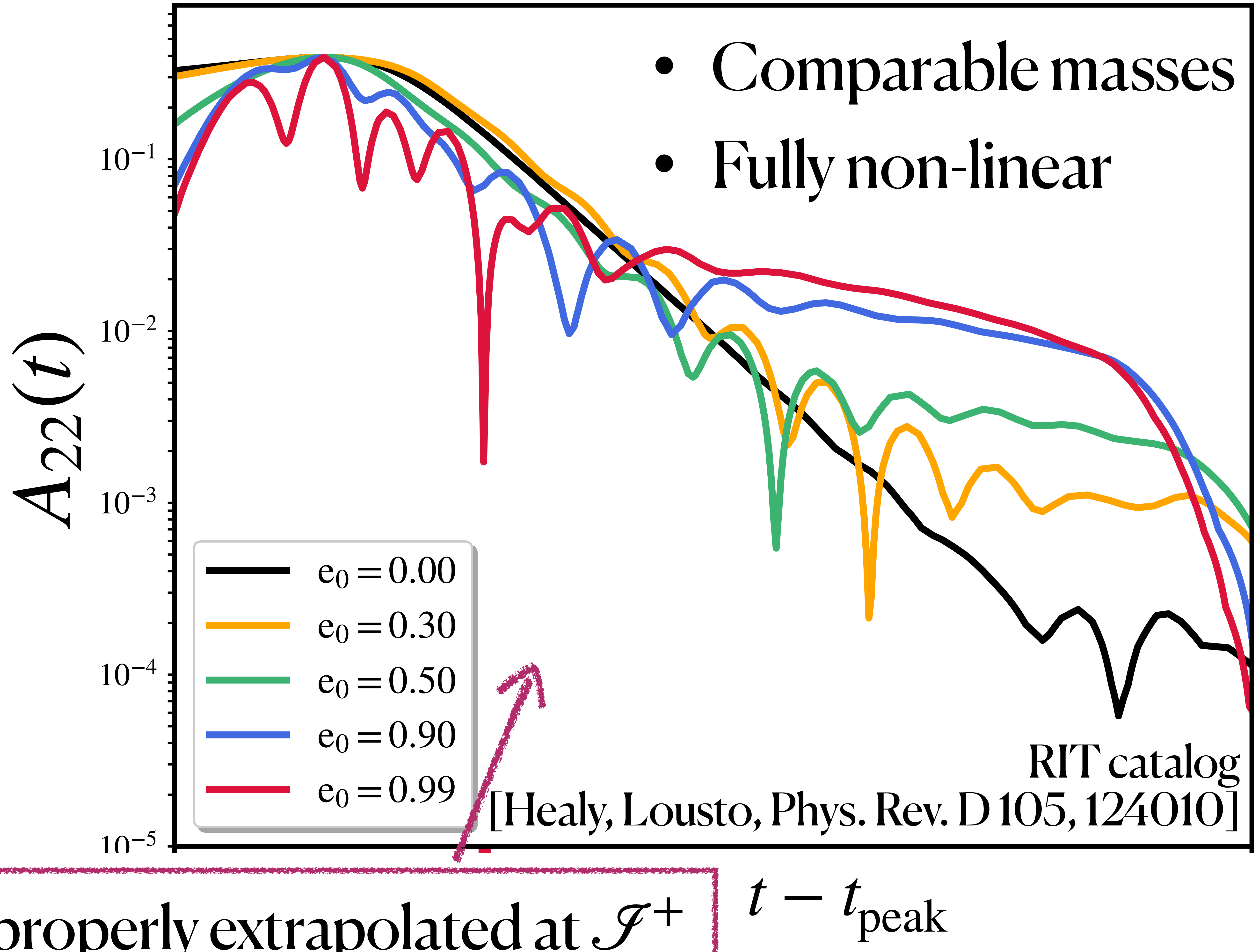
[Albanesi et al, Phys. Rev. D 108, 084037]

- Test-particle
- Perturbative results



[Carullo and **De Amicis**, 2310.12968]

- Comparable masses
- Fully non-linear



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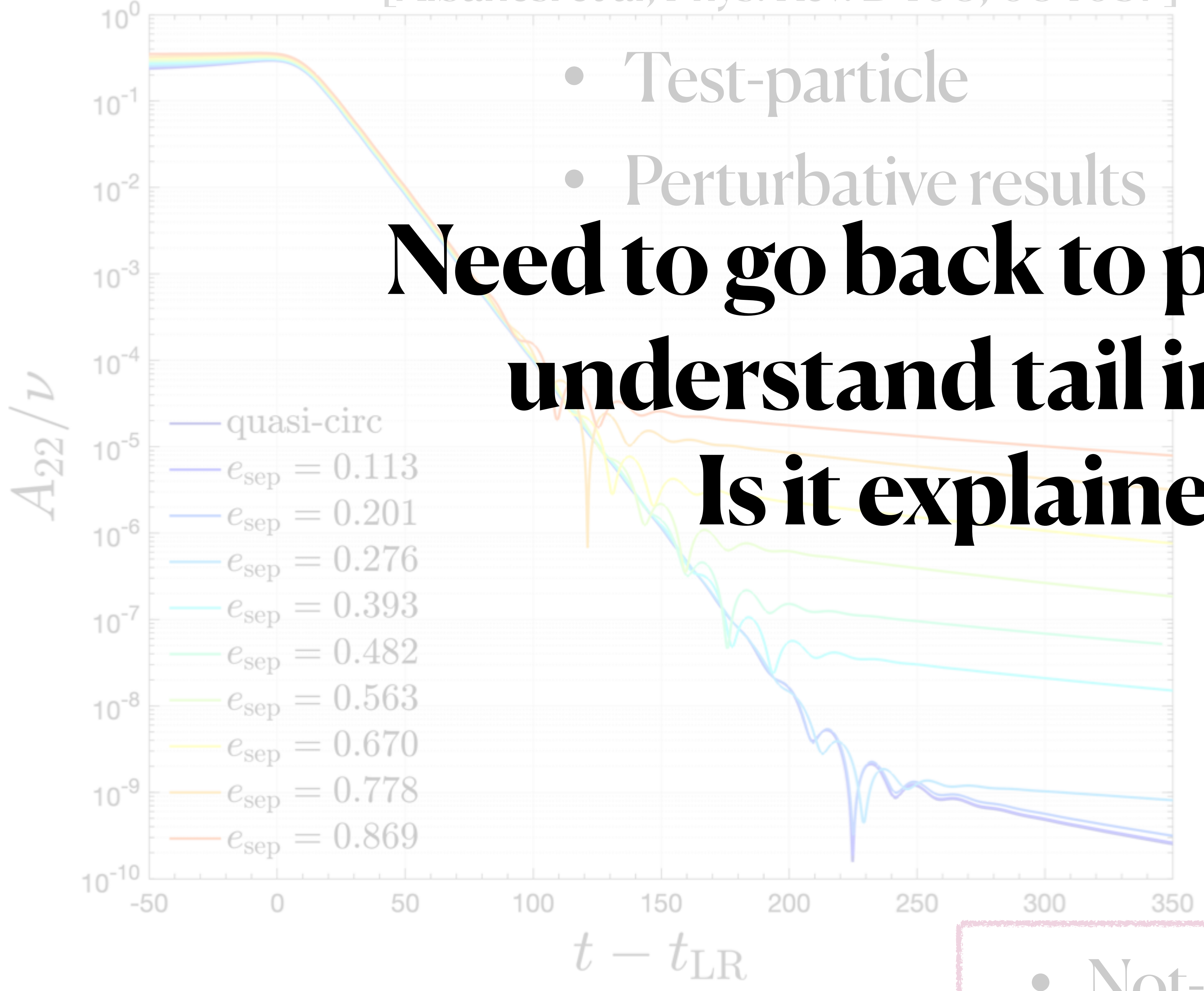

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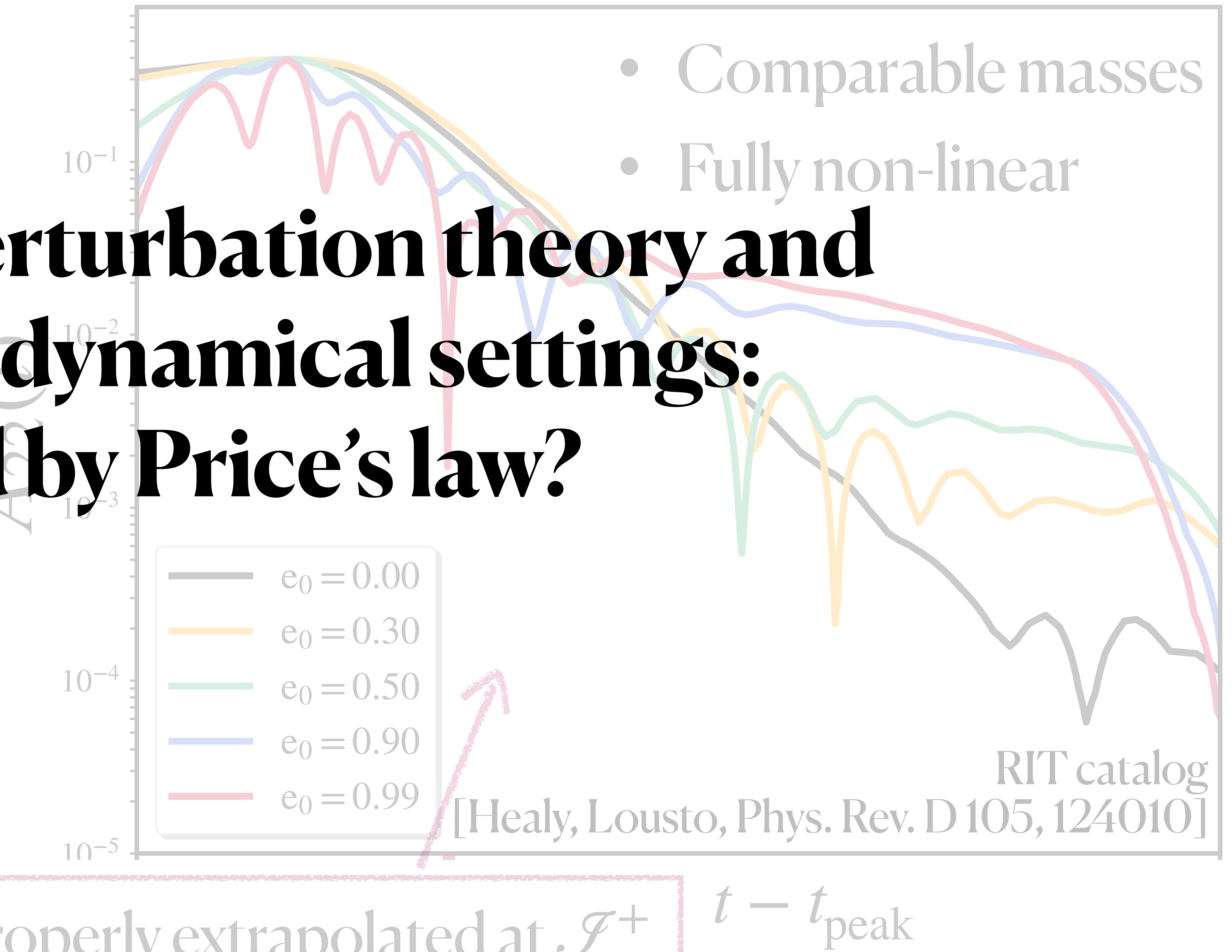

- Not-properly extrapolated at $\mathscr{I}^+$
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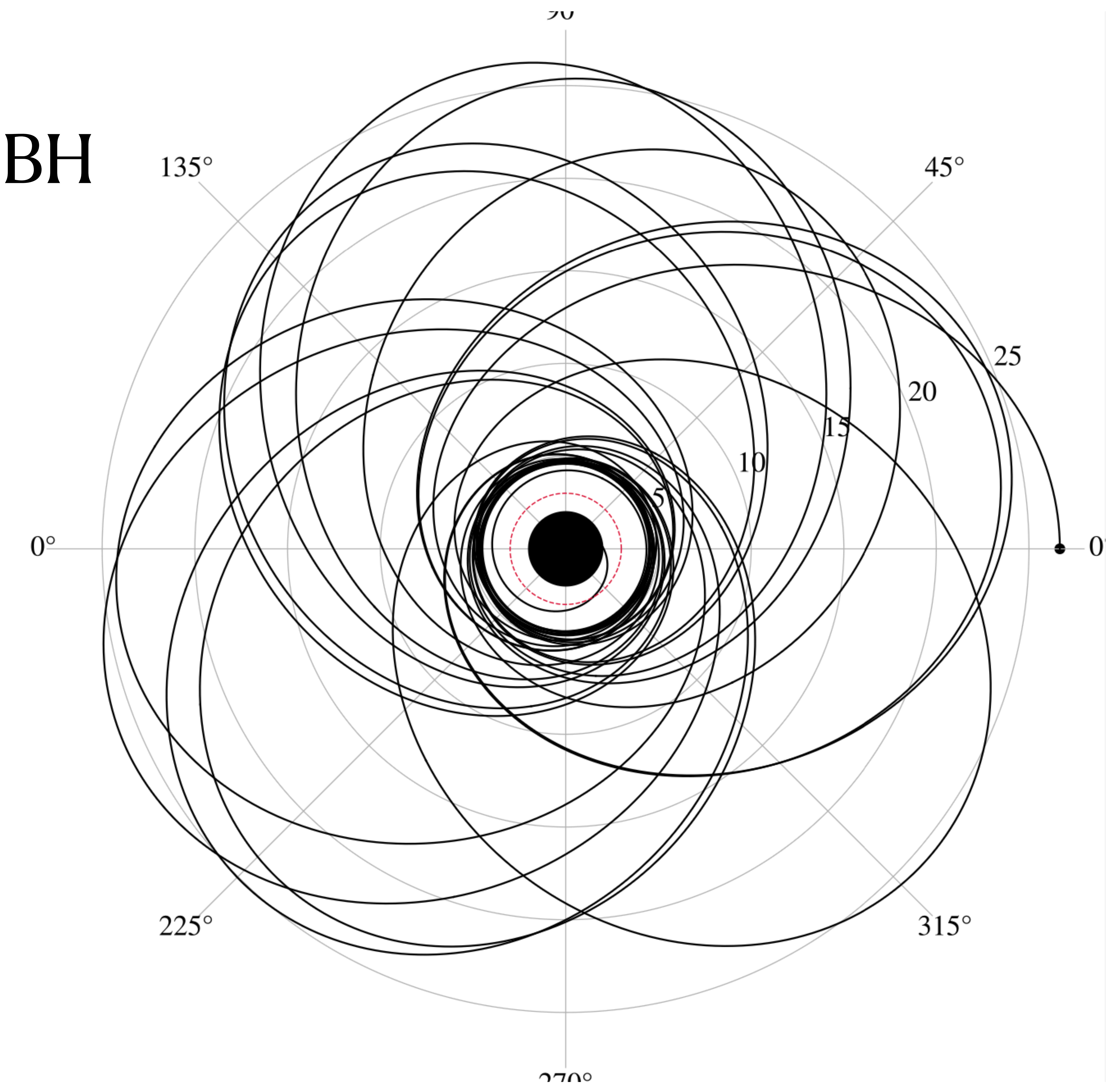
Need to go back to perturbation theory and understand tail in dynamical settings: Is it explained by Price's law?

Framework

- Test particle μ infalling in a Schwarzschild BH

- Perturbation theory



Framework

- Test particle μ infalling in a Schwarzschild BH
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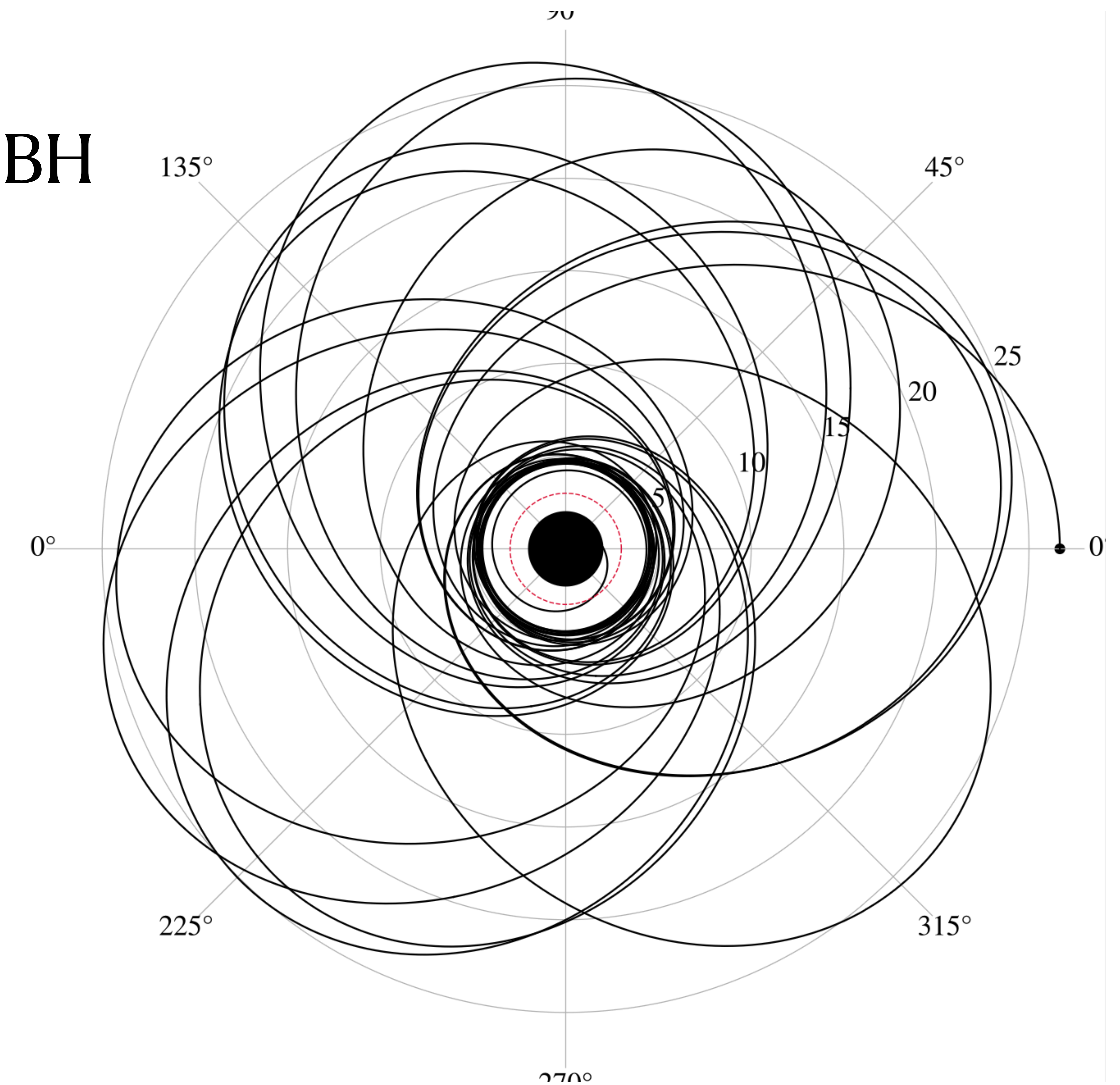
- Signal computed at scri+ (null infinity)



Framework

- Test particle μ infalling in a Schwarzschild BH

- Perturbation theory

- Signal computed at scri+ (null infinity)

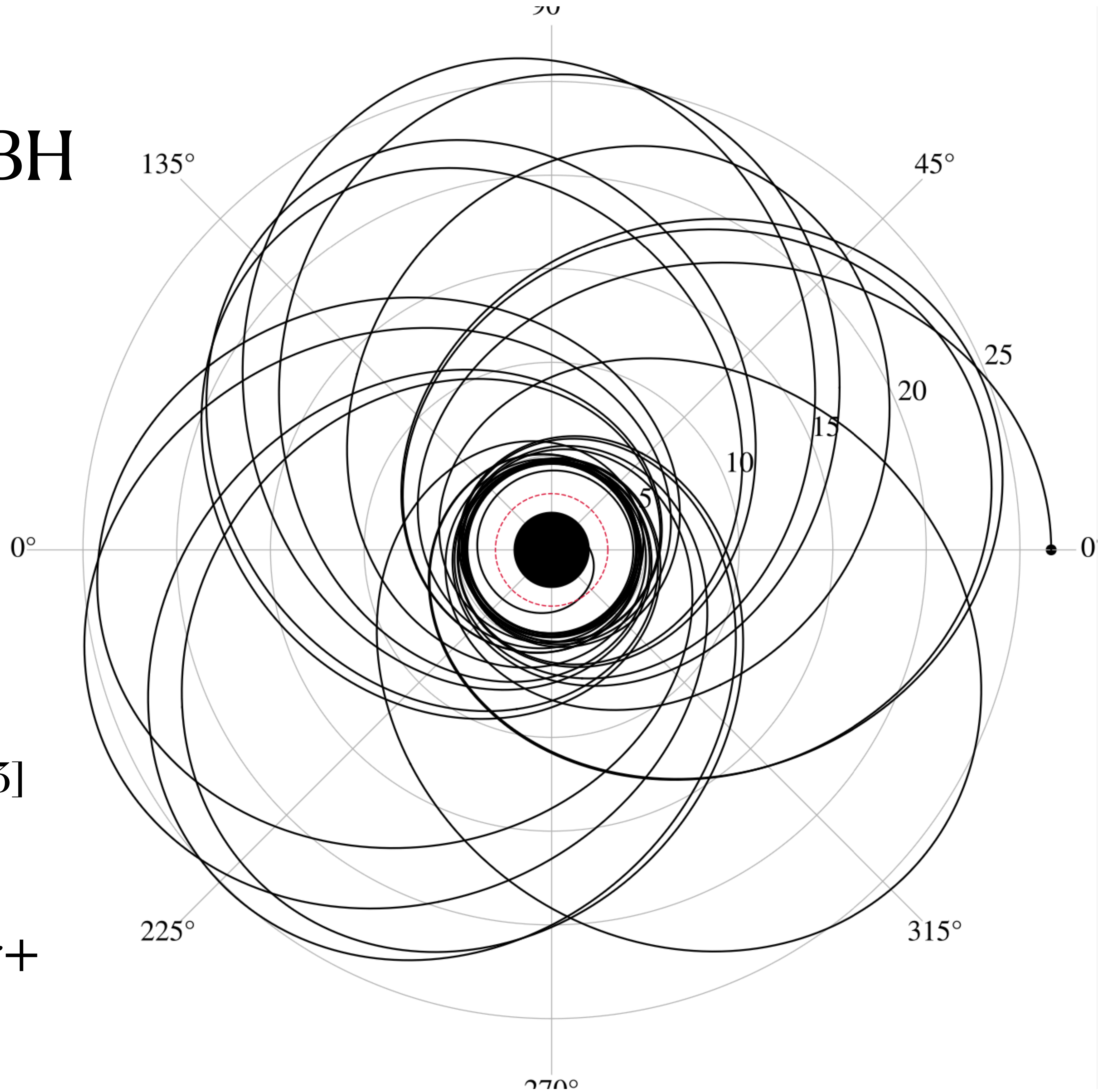
  - As observed by real detectors

    [Zenginoglu, Class.Quant.Grav. 25 (2008) 175013]

  - Price's law:

    - $\Psi_{\ell m} \propto \dfrac{1}{\tau^{\ell+2}}, \quad \tau \equiv t - r_* \;$ at $\mathscr{I}^+$

    - $\Psi_{\ell m} \propto \dfrac{1}{t^{2\ell+3}}$ at finite distance $\longrightarrow$ Suppressed!



[Price, Phys. Rev. D 5, 2419]
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Numerical evolutions

$$\left[\partial_t^2 - \partial_{r_*}^2 + V_{\ell m}^{(e/o)}(r_*)\right] \Psi_{\ell m}^{(e/o)}(t, r_*) = S_{\ell m}^{(e/o)}(t, r)$$

$$\Psi_{\ell m}^{(e/o)}(t = 0, r) = \partial_t \Psi_{\ell m}^{(e/o)}(t = 0, r) = 0$$

\+ **Hamiltonian equations of motion** for the trajectory, driven **radiation-reaction**

[Chiaramello and Nagar, Phys. Rev. D 101, 101501 (2020)]
[Albanesi, Nagar, Bernuzzi, Phys. Rev. D 104, 024067 (2021)]

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- **Hyperboloidal layer** over which r_* is **compatified**
- Compute the radiative signal at $\mathscr{I}^+$

RWZhyp code:
[Bernuzzi and Nagar, Phys. Rev. D 81, 084056(2010)]
[Bernuzzi, Nagar and Zenginoglu, Phys. Rev. D 84, 084026(2011)]

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Regge-Wheeler/
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Most general solution:

$$\Psi_{\ell m}(\tau, \rho_+) = \int_{T_{in}}^{\tau - \rho_+} dt' \int dr' S_{\ell m}(t', r')\, G_\ell(\tau, t'; r', \rho_+)$$

$\rho_+ \equiv$ location of $\mathscr{I}^+$ in the compactified coordinate

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Price's law propagator

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Analytical model

$$S_{\ell m}(t,r) = f_{\ell m}(t,r)\delta(r - r(t)) + g_{\ell m}(t,r)\partial_r\delta(r - r(t))$$

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$$\Psi_{\ell m}(\tau,\rho_+) = \frac{(-1)^\ell 2^{\ell+1}\ell!(\ell+1)!}{(2\ell+1)!}\int_{T_{\text{in}}}^{\tau-\rho_+} dt' \frac{r^\ell(t')\left[r\left(f_{\ell m}(t') - \partial_r g_{\ell m}(t')\right) - (\ell+1)\, g_{\ell m}(t')\right]_{r=r(t')}}{\left(\tau-\rho_+-t'\right)^{\ell+2}}$$

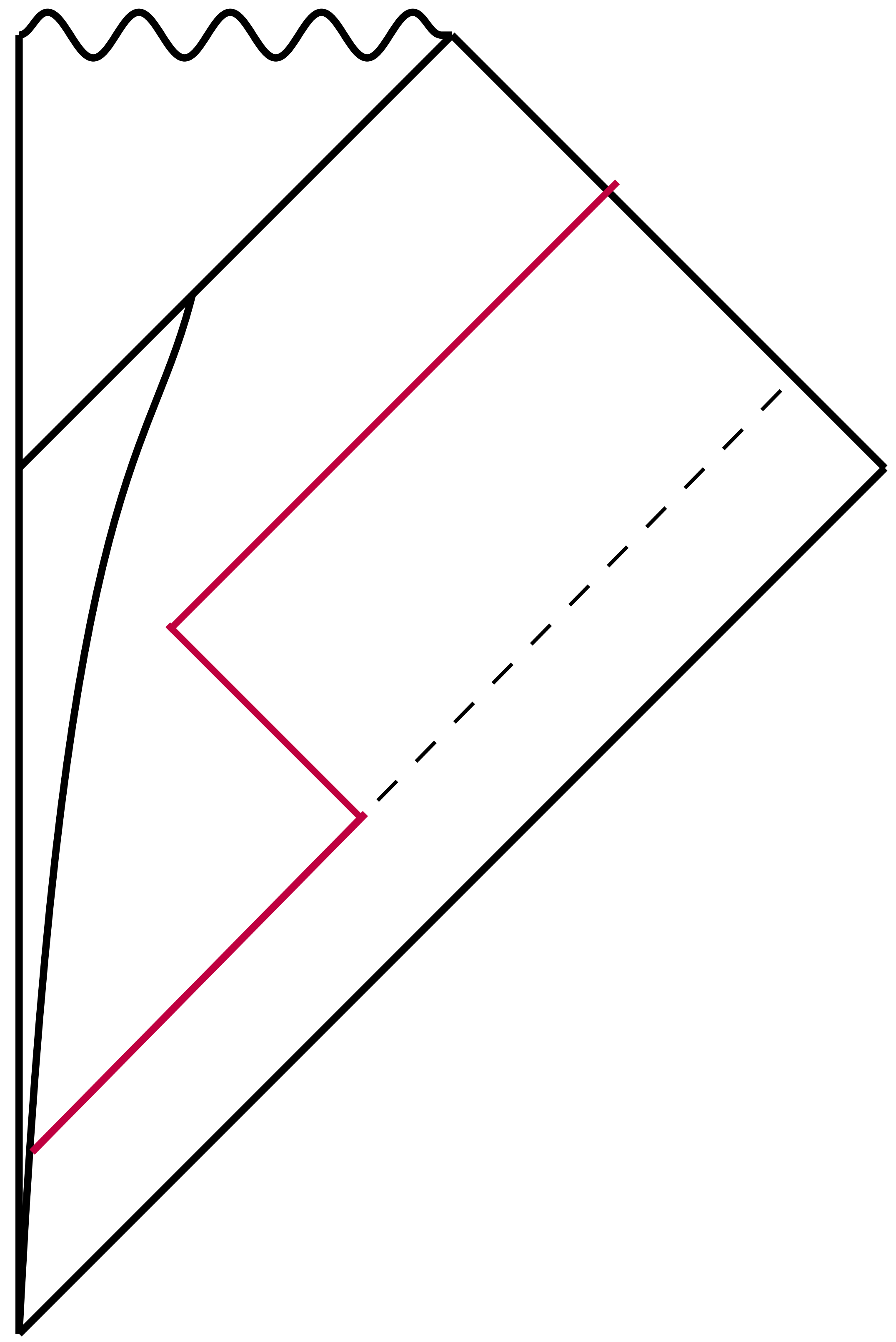
[De Amicis, Albanesi and Carullo, 2406.17018]

Analytical model

Analytical integral form of the tail:

$$\Psi_{\ell m}(\tau,\rho_+) = \frac{(-1)^{\ell} 2^{\ell+1} \ell!(\ell+1)!}{(2\ell+1)!} \int_{T_{\rm in}}^{\tau-\rho_+} dt' \frac{r^{\ell}(t')\left[r\left(f_{\ell m}(t') - \partial_r g_{\ell m}(t')\right) - \left(\ell+1\right) g_{\ell m}(t')\right]_{r=r(t')}}{\left(\tau - \rho_+ - t'\right)^{\ell+2}}$$

- Tail as a **memory effect**



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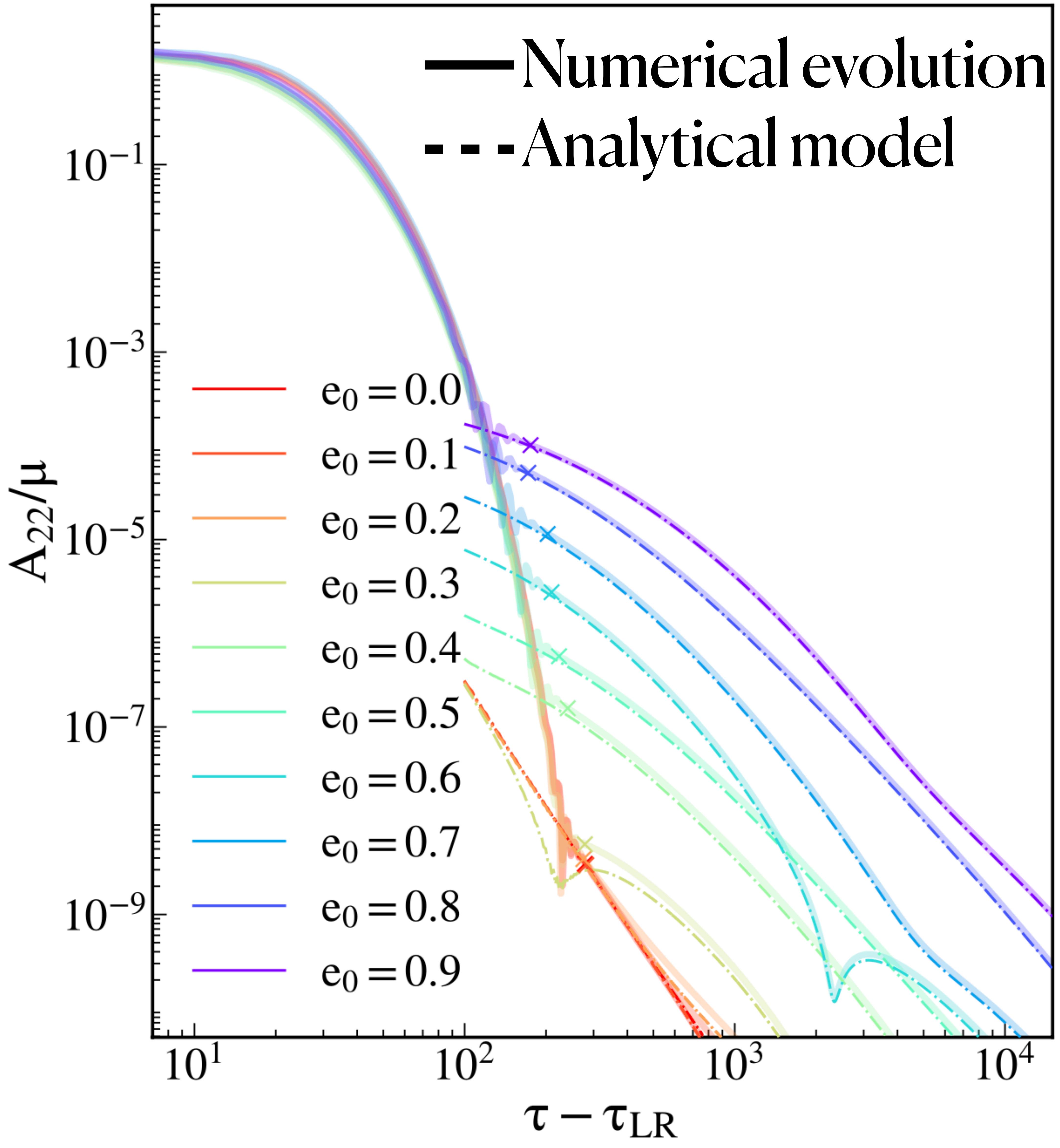
- Tail as a **memory effect**
- **Not** an **exact power-law** behavior

[De Amicis, Albanesi and Carullo, 2406.17018]

Model vs numerical evolutions: eccentric orbits



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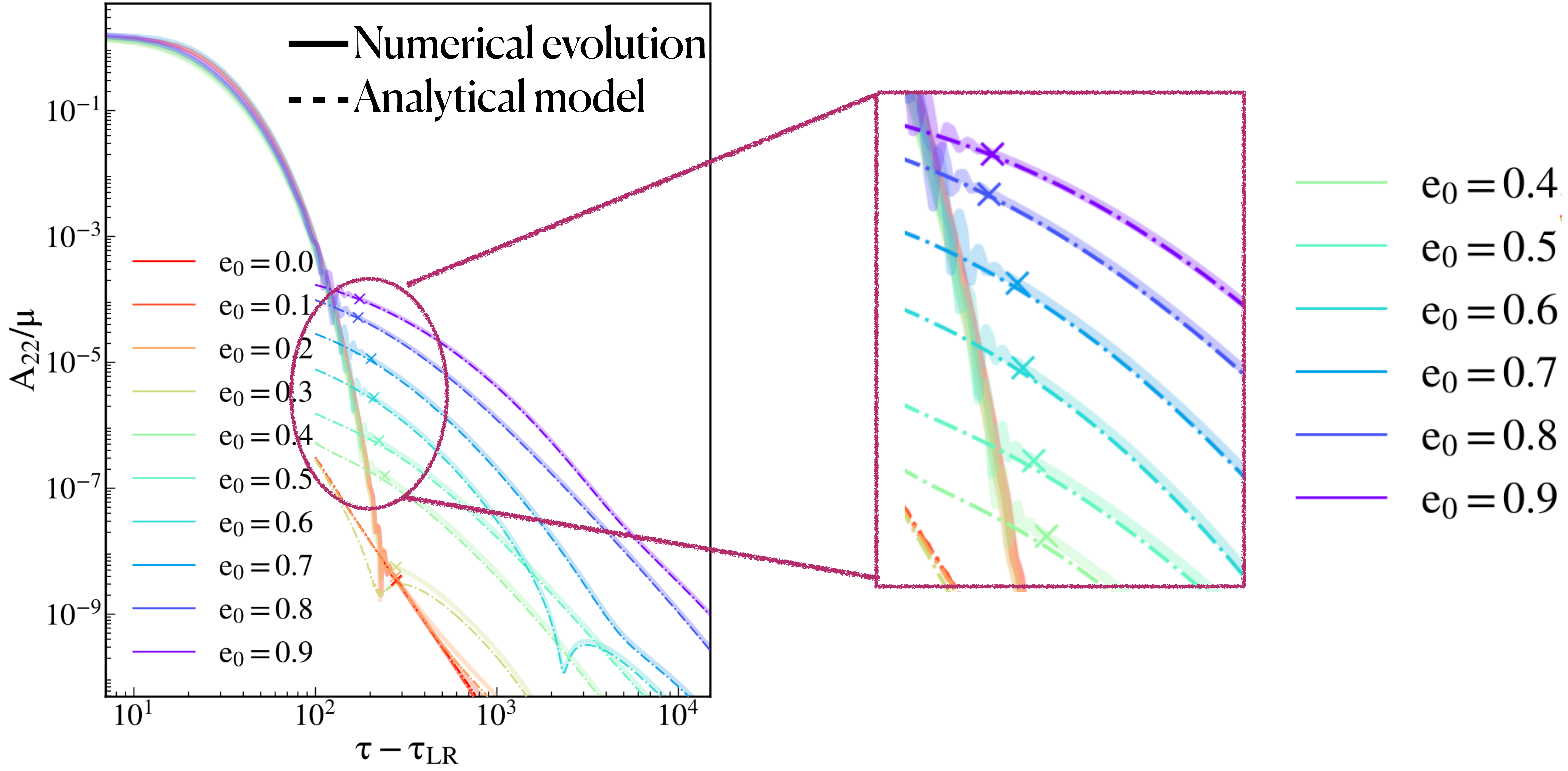

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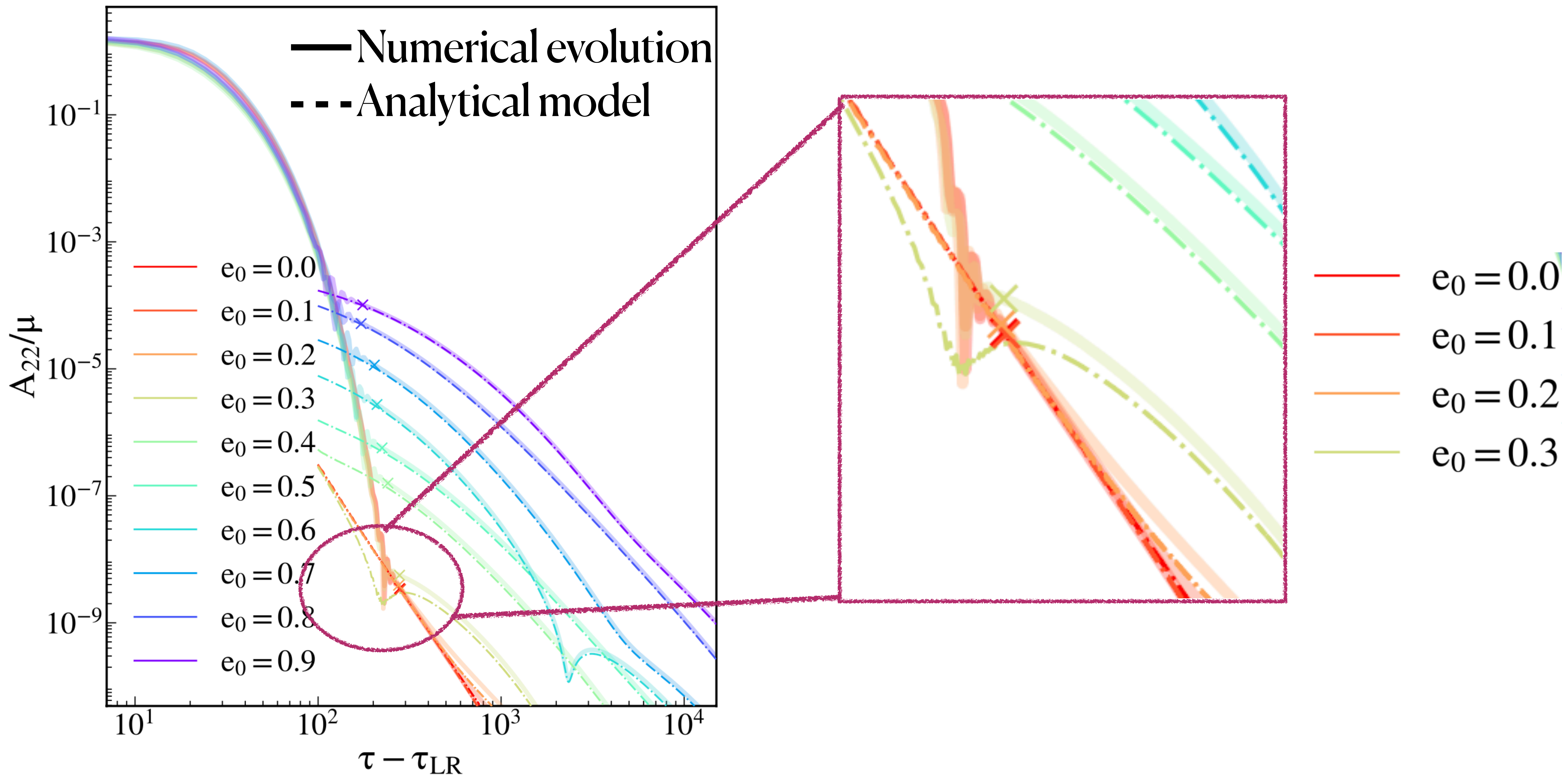

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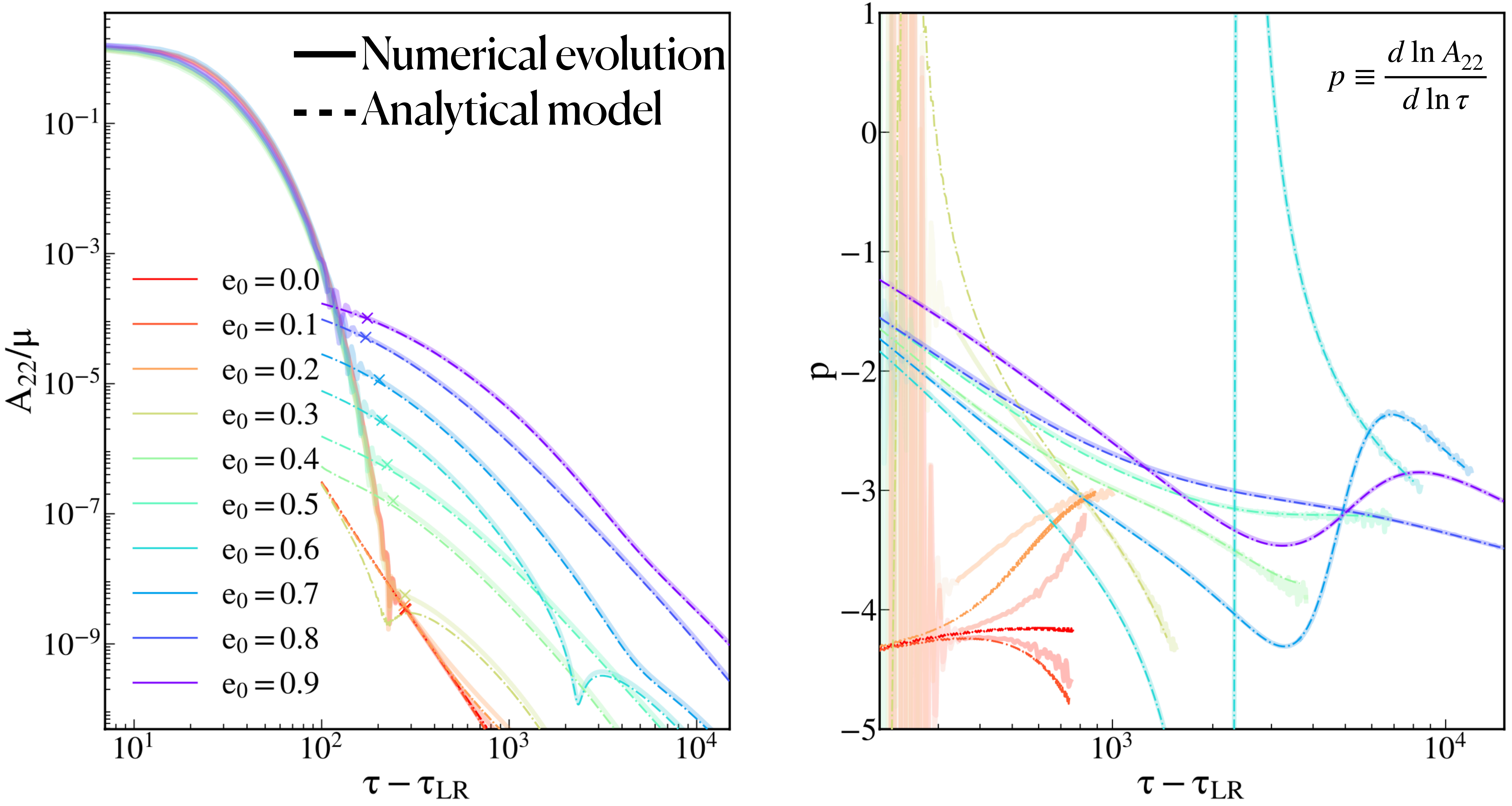

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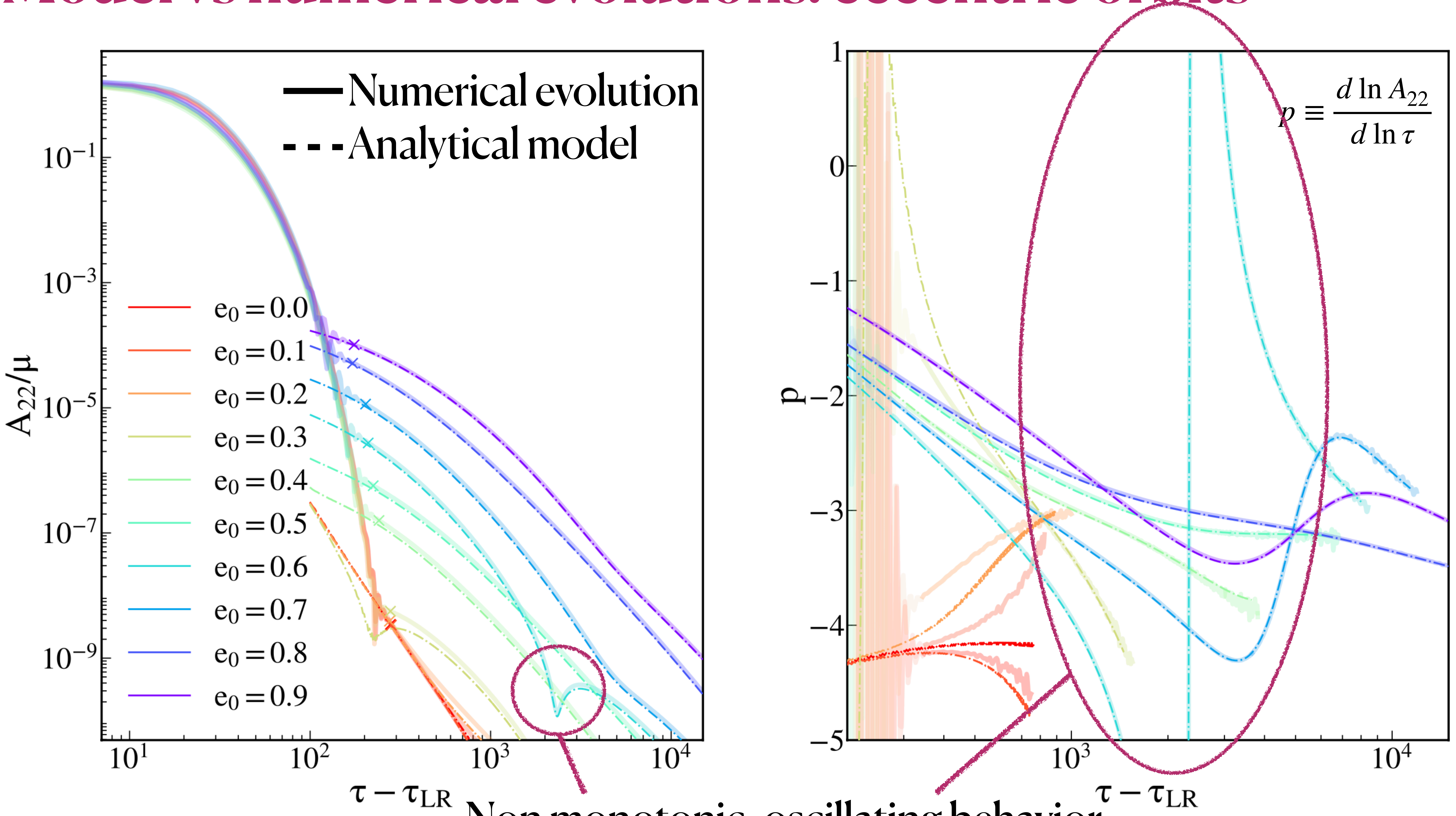

Non monotonic, oscillating behavior

Take a breath

- Integral model for tail in EMR, as a memory effect ✓
- Tail exponent is in general non monotonic
- Tail amplitude is enhanced by eccentricity

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Tail as superposition of power-laws

$\tau - \rho_+ \gg t_{\text{in}}, t_f$

t_{in} = initial time

t_f = common horizon

Tail as superposition of power-laws

$$\Psi_{\ell m}(\tau,\rho_+) = \frac{c_\ell}{\tau^{\ell+2}} \int_{t_{\rm in}}^{t_f} dt'\ S_\ell(t') \left[1 + \sum_{n=1}^{\infty} \frac{(\ell+1+n)!}{n!(\ell+1)!} \left(\frac{t'+\rho_+}{\tau} \right)^n \right]$$

$\tau - \rho_+ \gg t_{\rm in}, t_f$

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- Superposition of power-laws
- Slowest decay is Price's law

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Tail as superposition of power-laws

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- Superposition of power-laws
- Slowest decay is Price's law
- **Excitation coefficient** of each power-law depends on:
  - **amount of history**

Tail as superposition of power-laws

$$\Psi_{\ell m}(\tau,\rho_+) = \frac{c_\ell}{\tau^{\ell+2}} \int_{t_{\rm in}}^{t_f} dt' \, S_\ell(t') \left[1 + \sum_{n=1}^{\infty} \frac{(\ell+1+n)!}{n!(\ell+1)!} \left(\frac{t'+\rho_+}{\tau} \right)^n \right]$$

$\tau - \rho_+ \gg t_{\rm in}, t_f$

- Superposition of power-laws
- Slowest decay is Price's law
- **Excitation coefficient** of each power-law depends on:
  - **amount of history**
  - **specific orbit**

$t_{\rm in}$ = initial time

t_f = common horizon

Tail as superposition of power-laws

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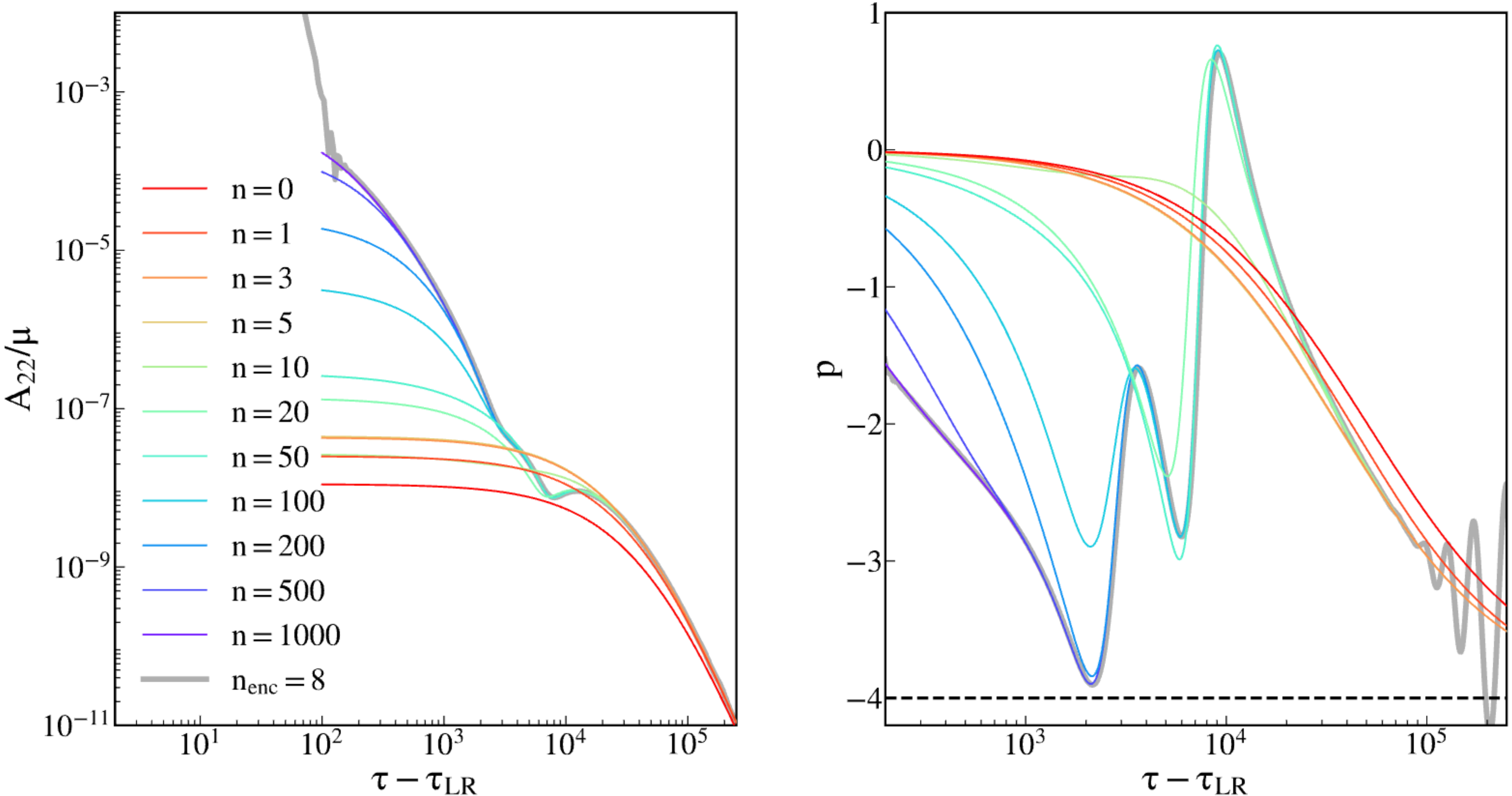

Marina De Amicis

Take a breath

- Integral model for tail in EMR, as a memory effect
- Tail as superposition of power laws $\tau^{-\ell-2-n}$, with $n \geq 0$
- Tail amplitude is enhanced by eccentricity

Enhancement with eccentricity

Isolate the part of the trajectory which determines the amplitude at the transition from QNMs to tail

Enhancement with eccentricity

$$\Psi_{\ell m}(\tau, \rho_+) = \frac{(-1)^{\ell} 2^{\ell+1} \ell!(\ell+1)!}{(2\ell+1)!} \int_{t_{\rm in}}^{\tau-\rho_+} dt' \frac{r^{\ell}(t')\Big[r\left(f_{\ell m}(t') - \partial_r g_{\ell m}(t')\right) - \left(\ell+1\right) g_{\ell m}(t')\Big]_{r=r(t')}}{\left(\tau-\rho_+-t'\right)^{\ell+2}}$$

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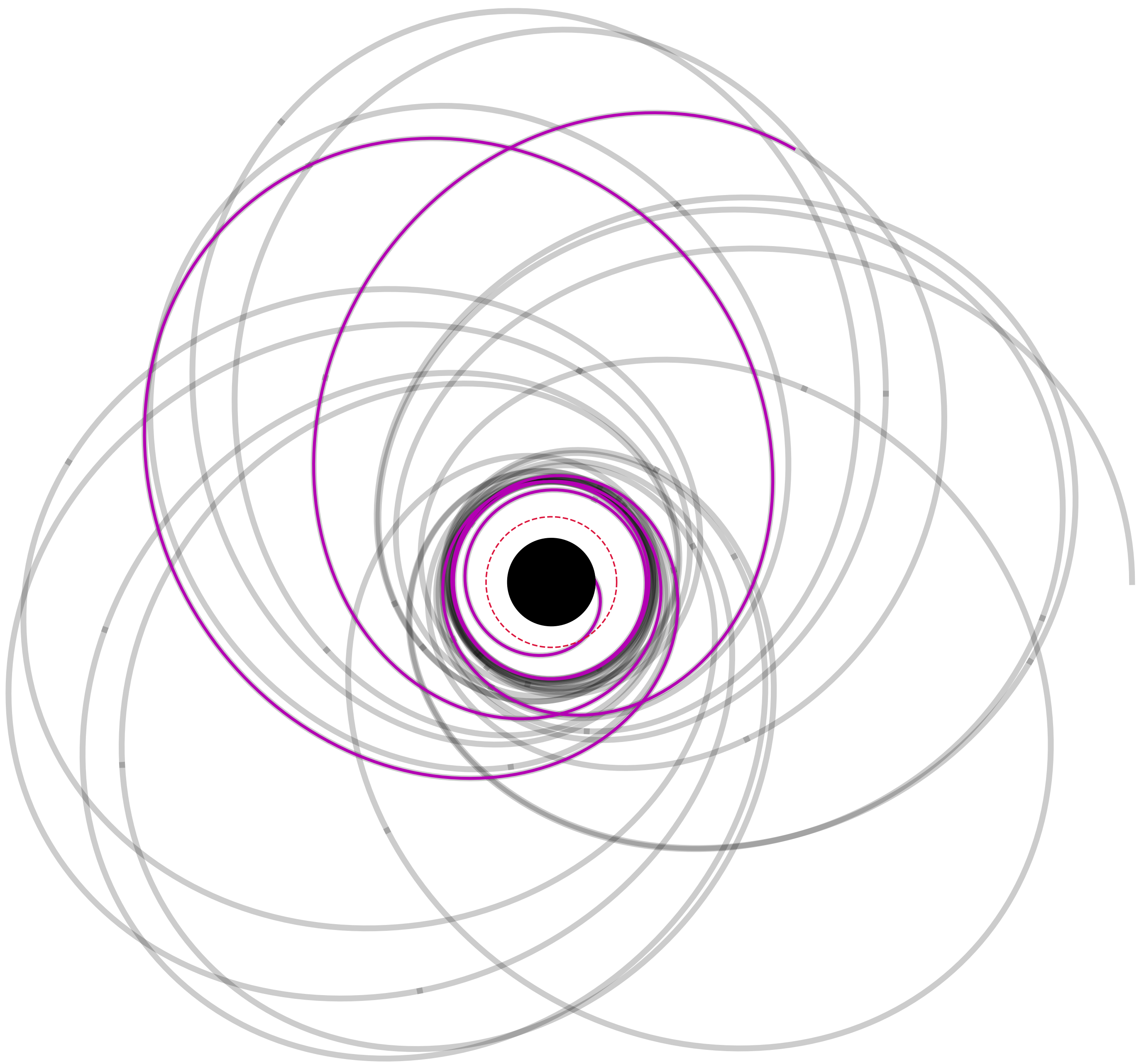

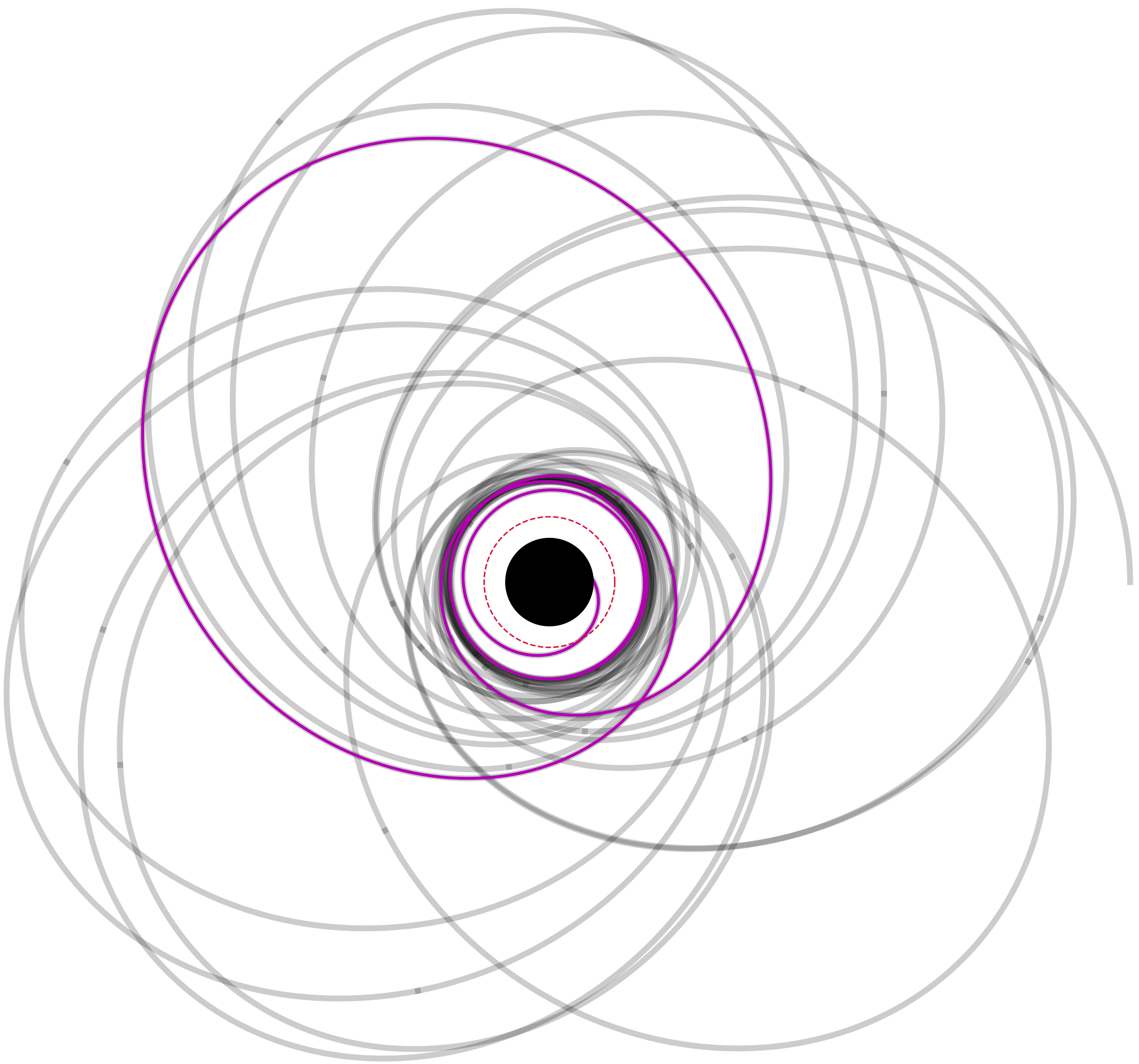

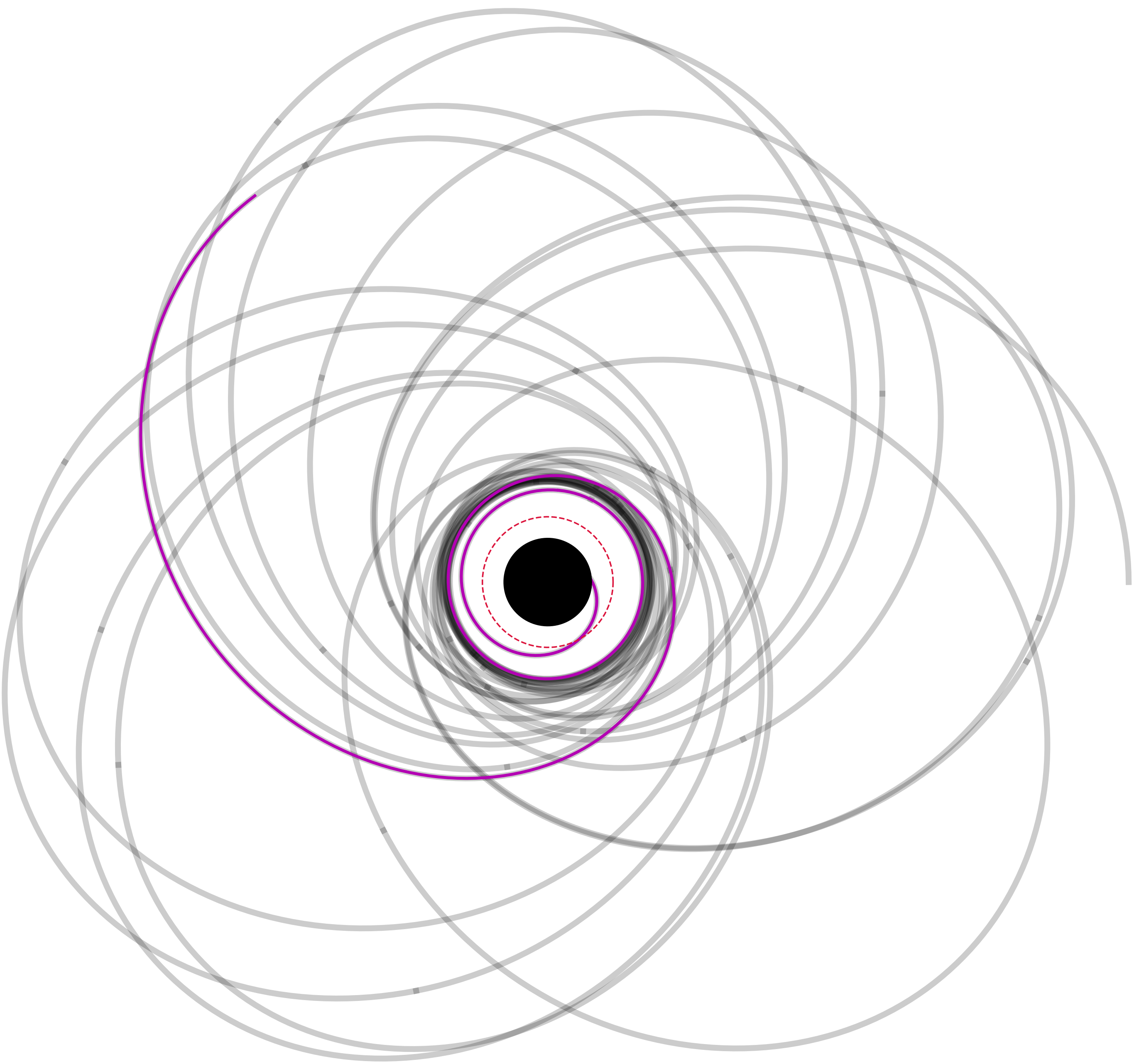

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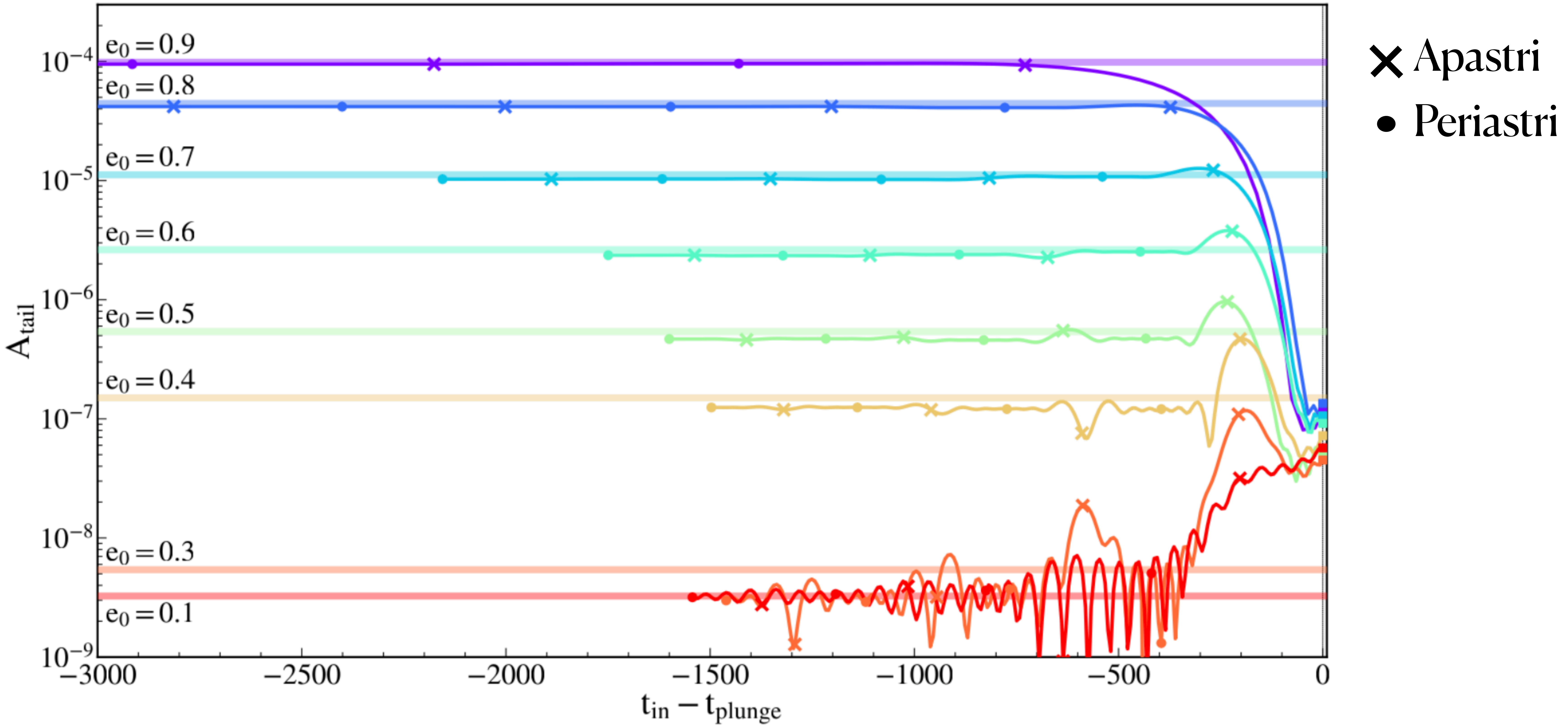

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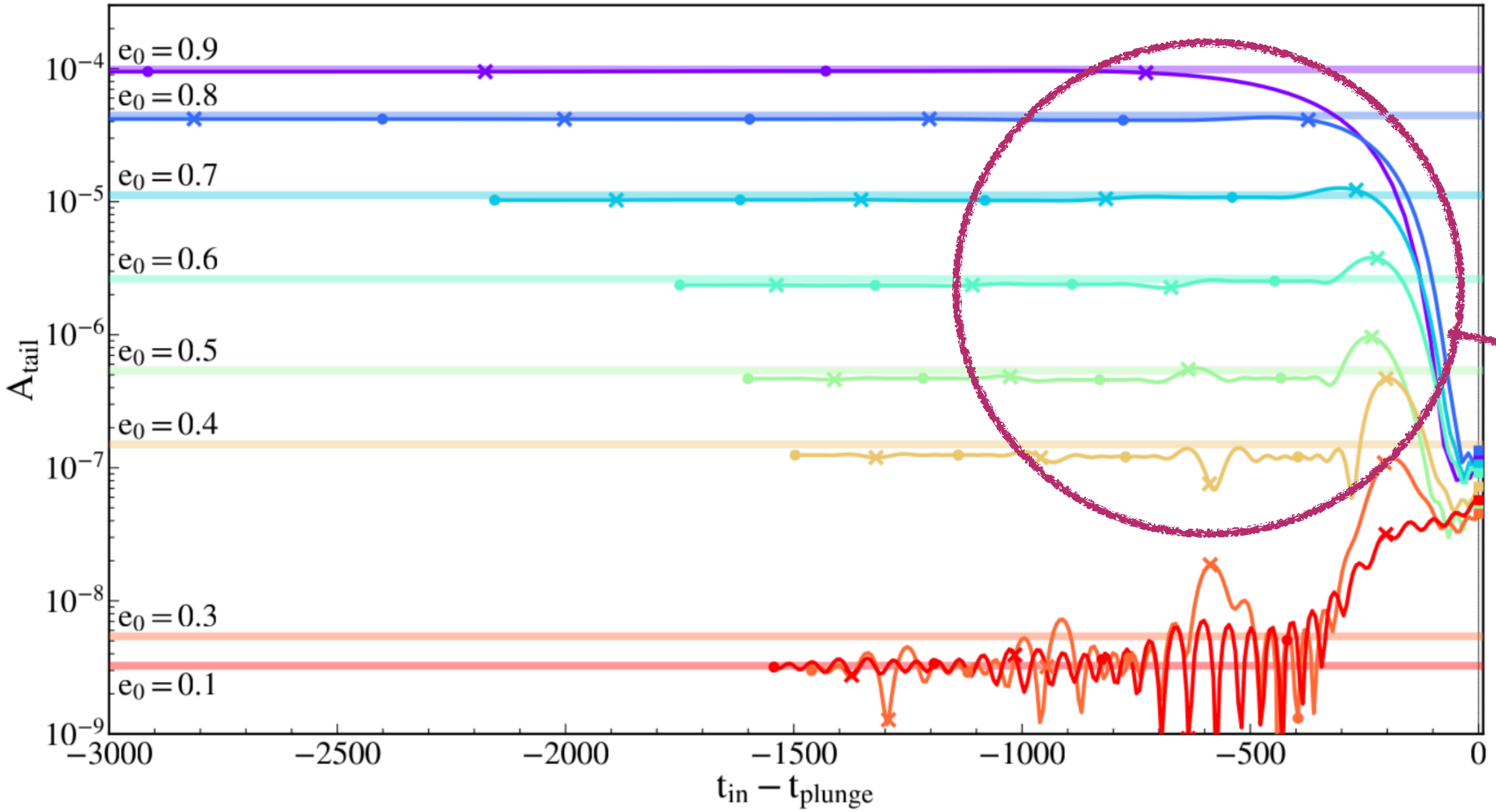

× Apastri

• Periastri

Tail amplitude:

- Determined by motion near last apastron

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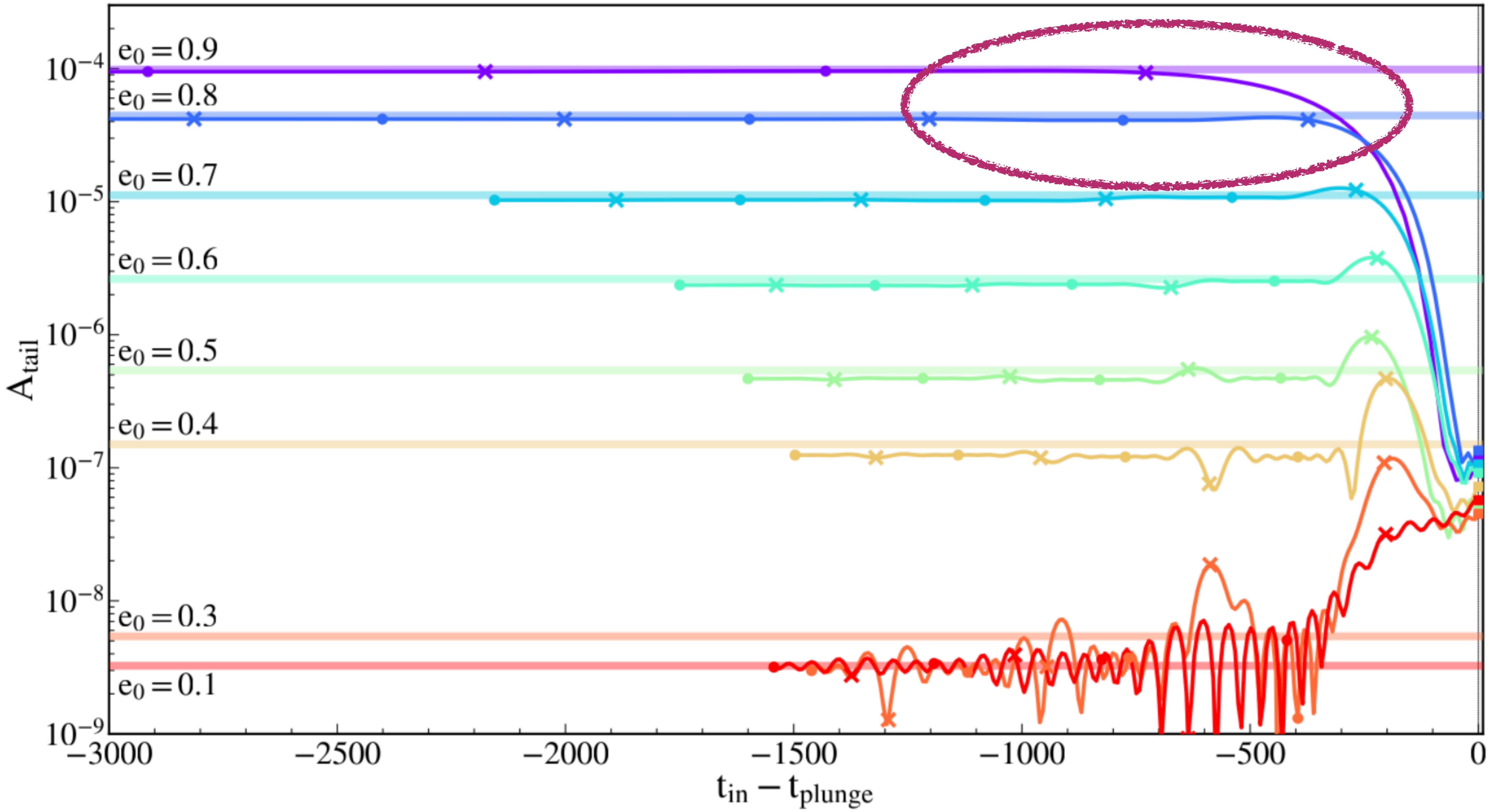

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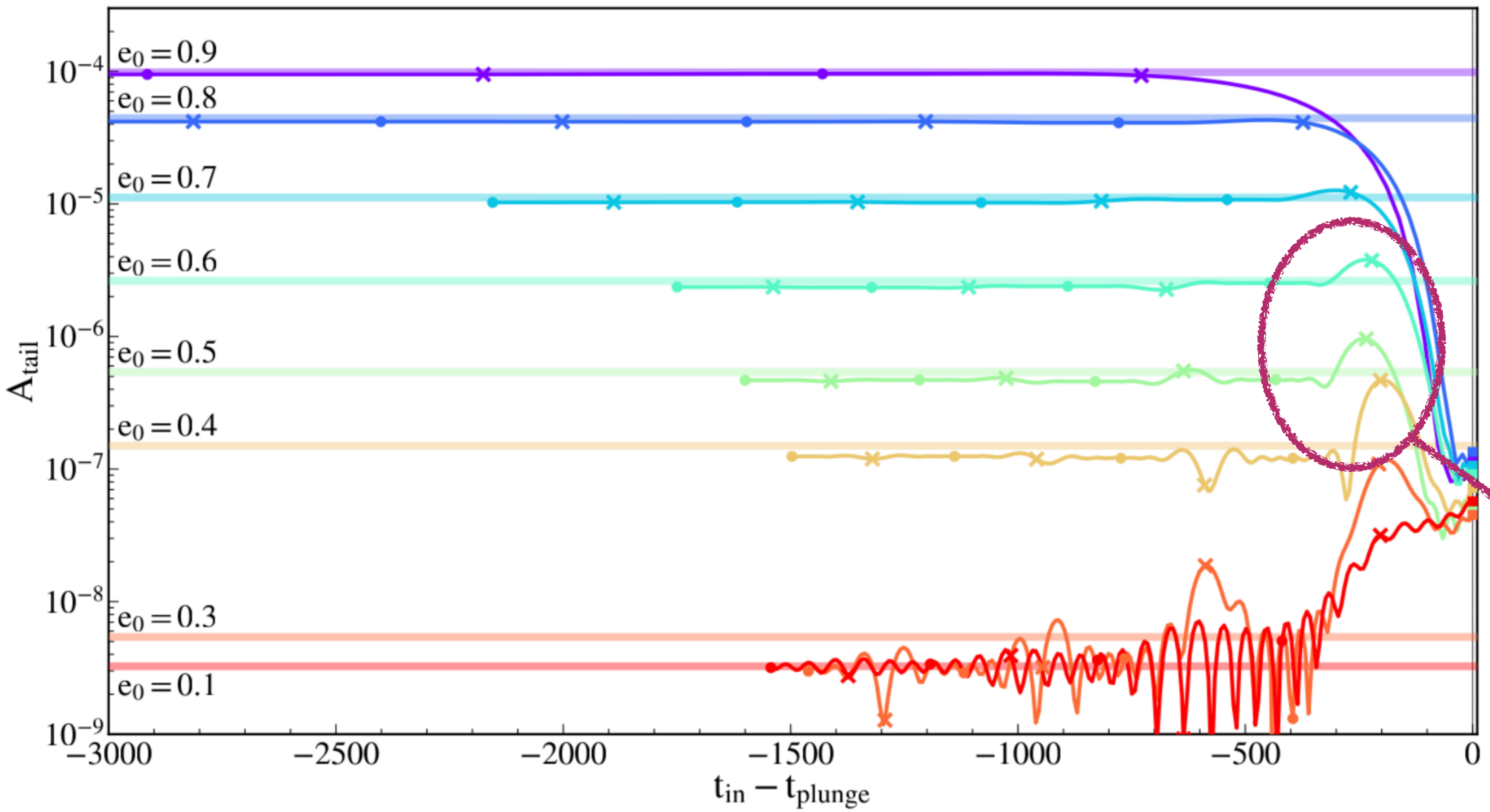

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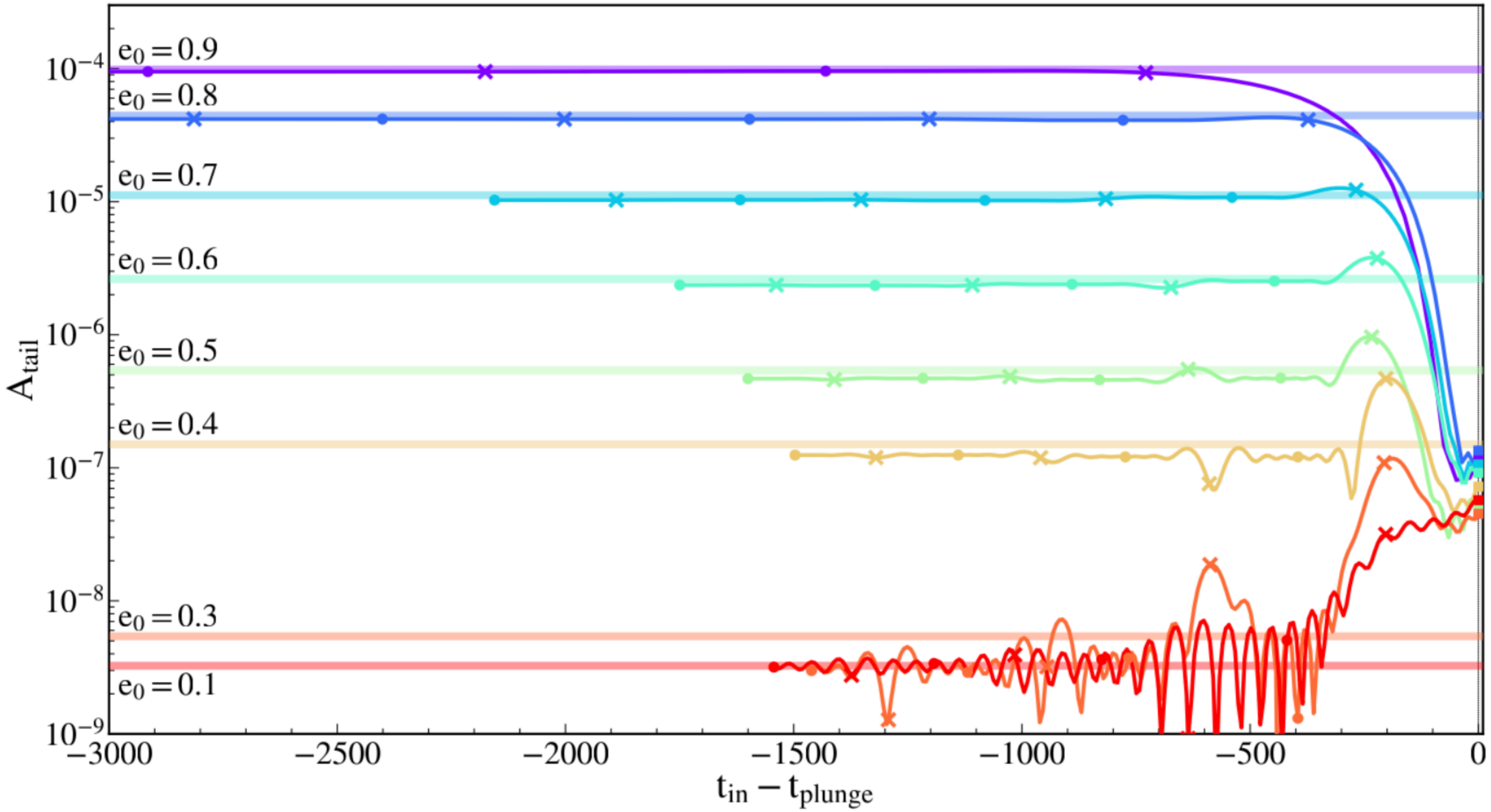

× Apastri

• Periastri

Tail amplitude:

- Determined by motion near last apastron → $r \gg M$, $p_\varphi / r \ll 1$
- Cancellation among in/outgoing motion

Enhancement with eccentricity

Expand in large r and small p_φ/r: $\Psi_{\ell m}(\tau,\rho_+) = \int_{t_{in}}^{t_f} dt' \frac{r^\ell(t') e^{-im\varphi(t')} P_{\ell m}(\cos\theta_0)}{(\tau - t' - \rho_+)^{\ell+2}} \cdot \left[a_1 - \frac{a_1}{2}\dot{r}^2 + a_2\dot{r}\frac{p_\varphi}{r} + \left(a_3 + \frac{a_1}{2}\right)\frac{p_\varphi^2}{r^2}\right]$

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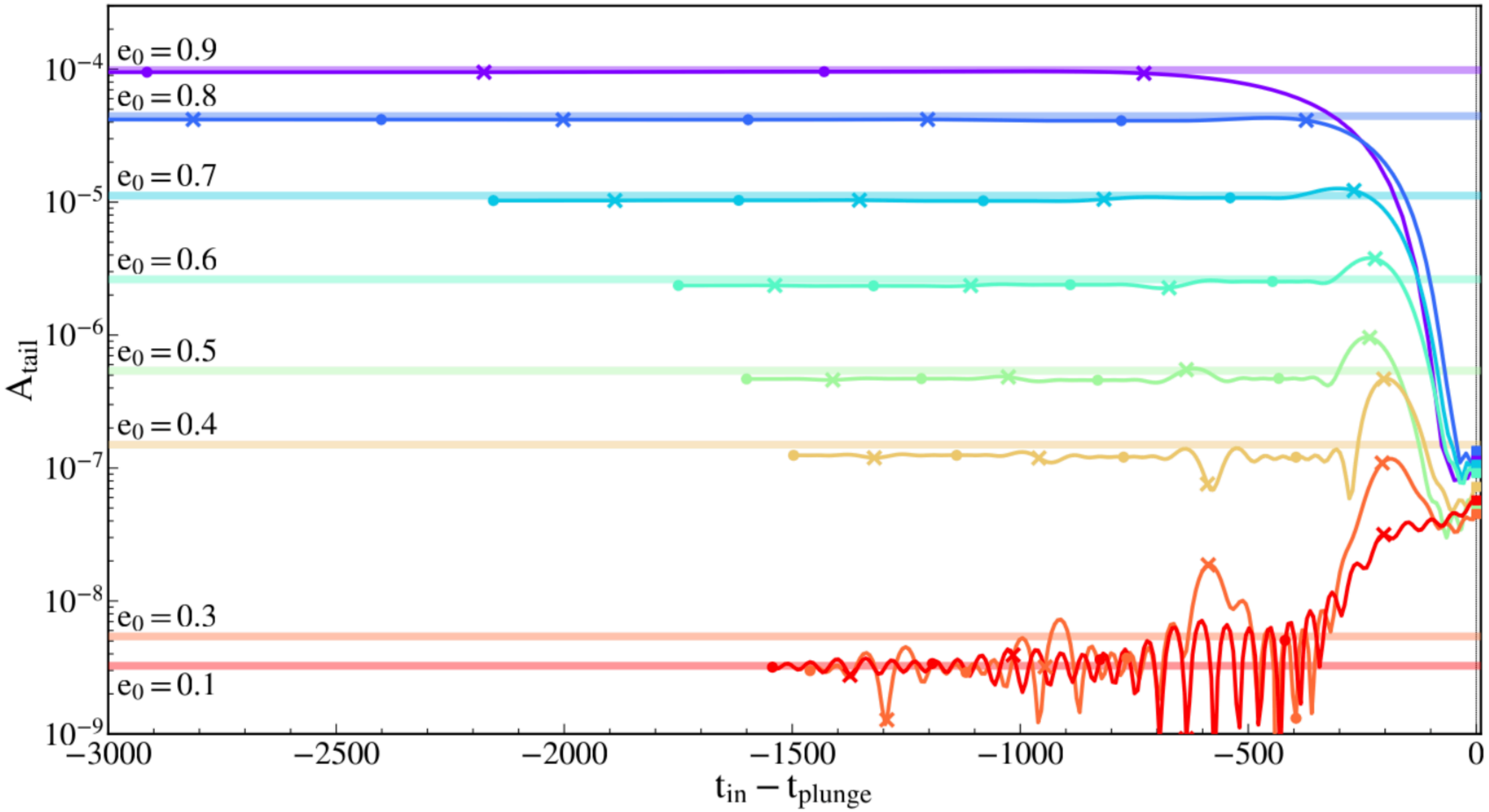

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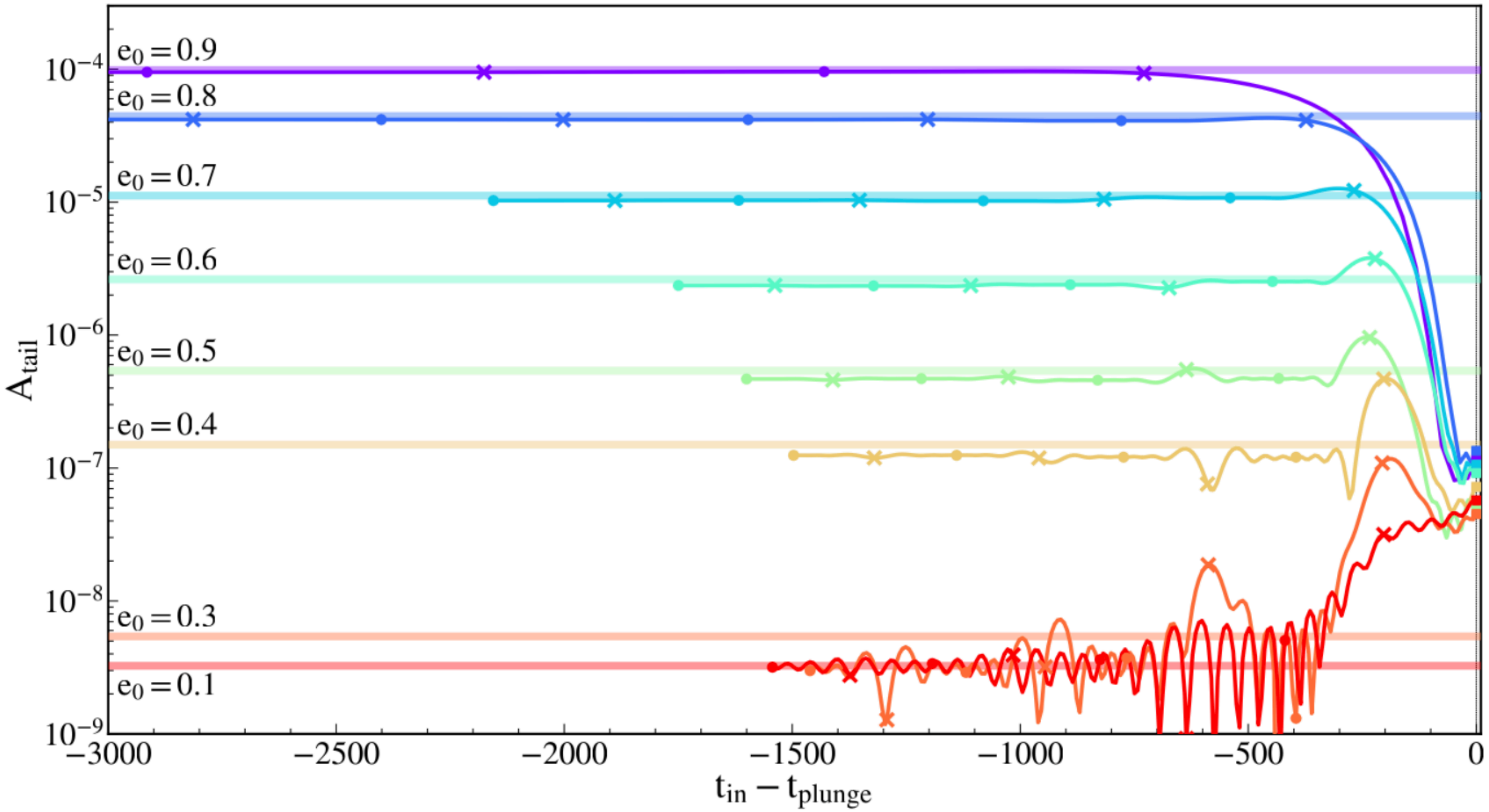

Oscillanting contribution can induce destructive interference

↓

- Tail maximied for radial infall!

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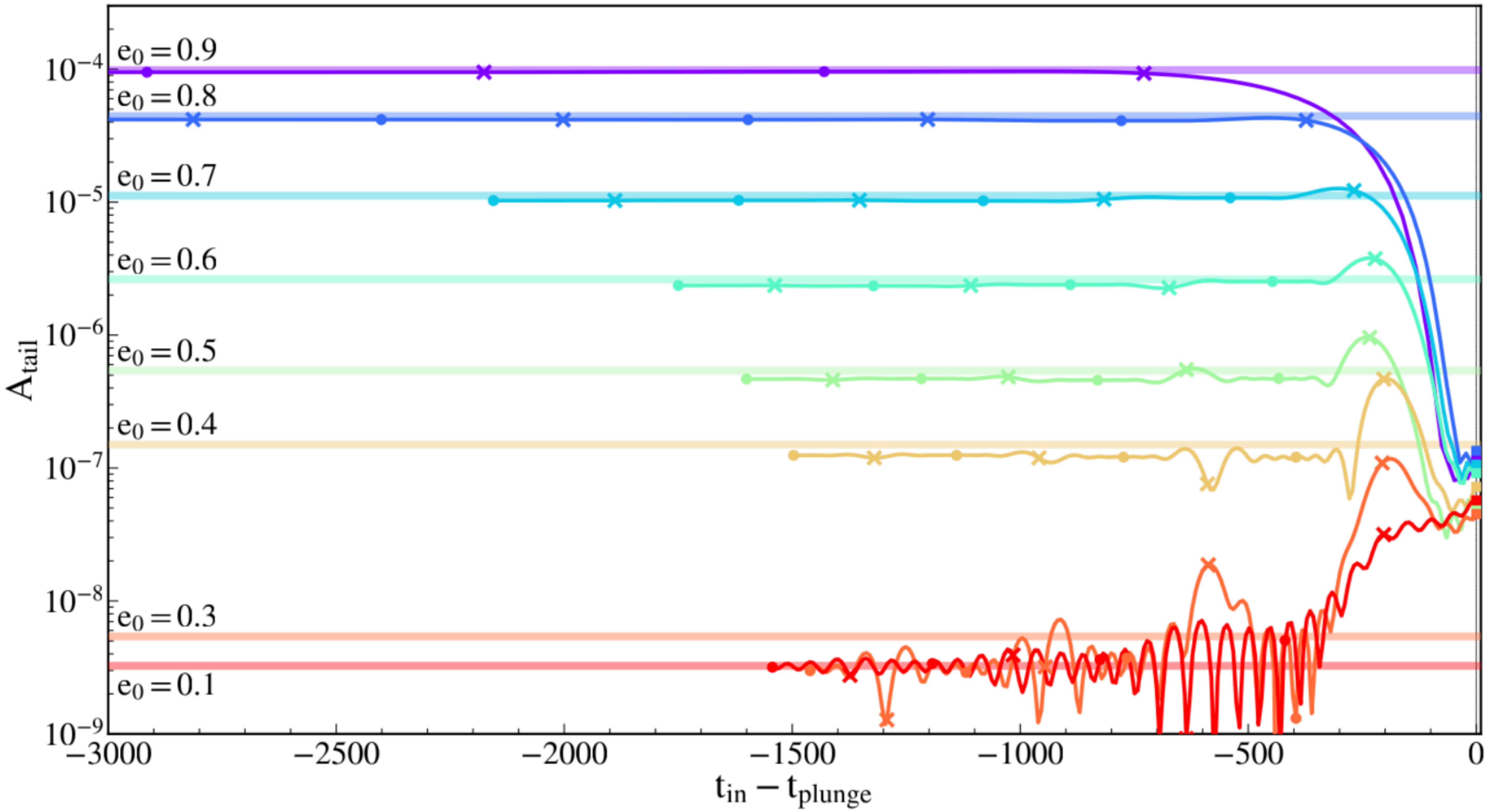

Oscillanting contribution can induce destructive interference

- Tail maximied for radial infall!
- Tail enhanced for $m = 0$ modes, even for quasi-circular binaries

Enhancement with eccentricity

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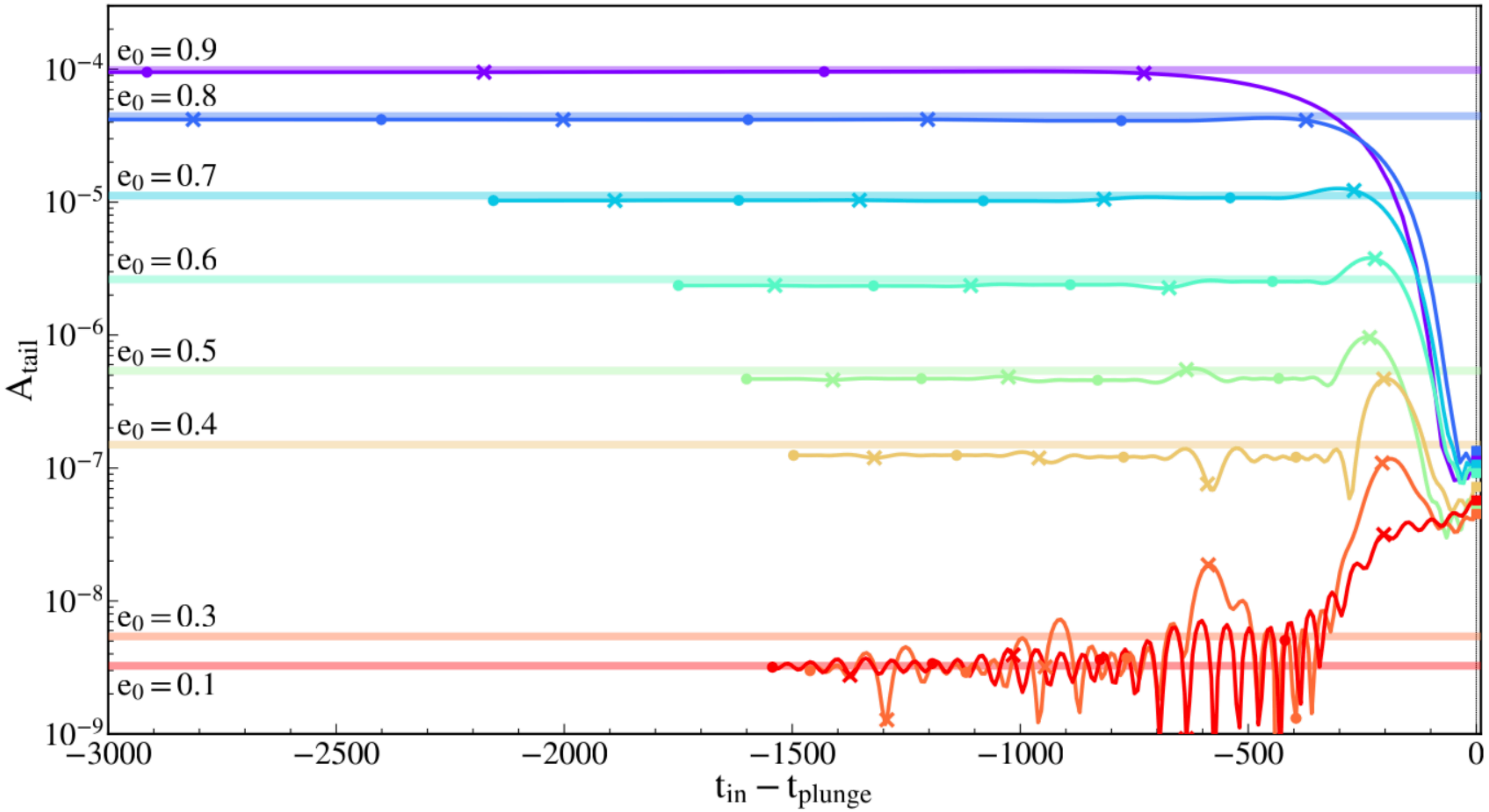

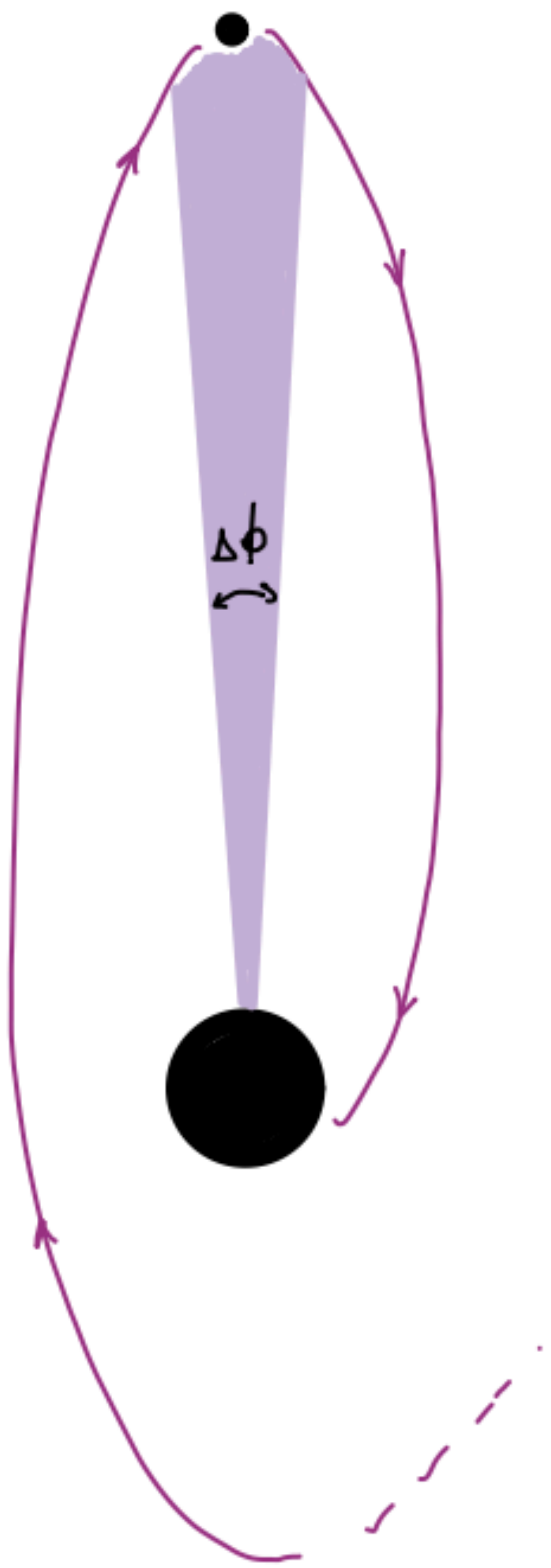

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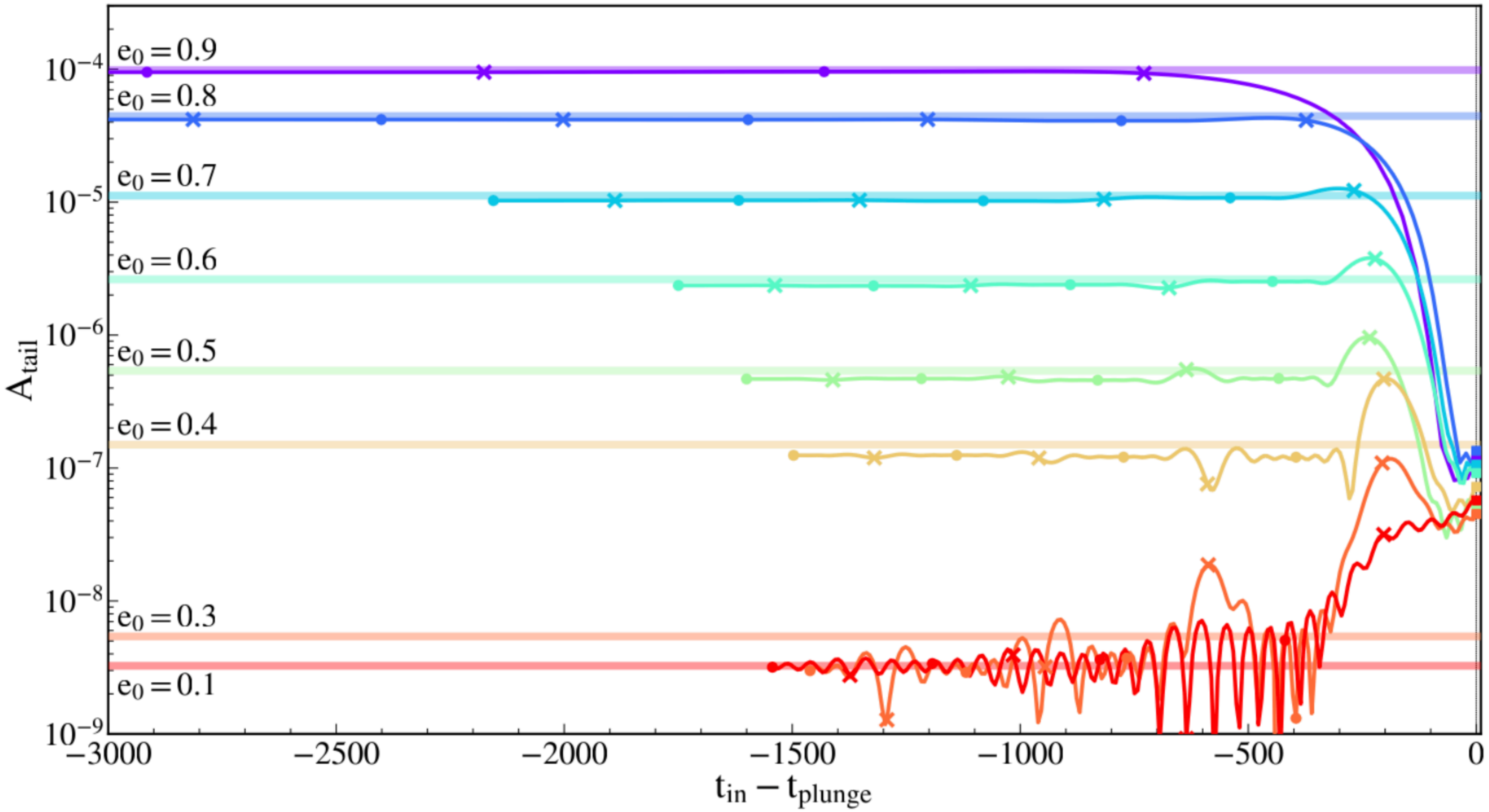

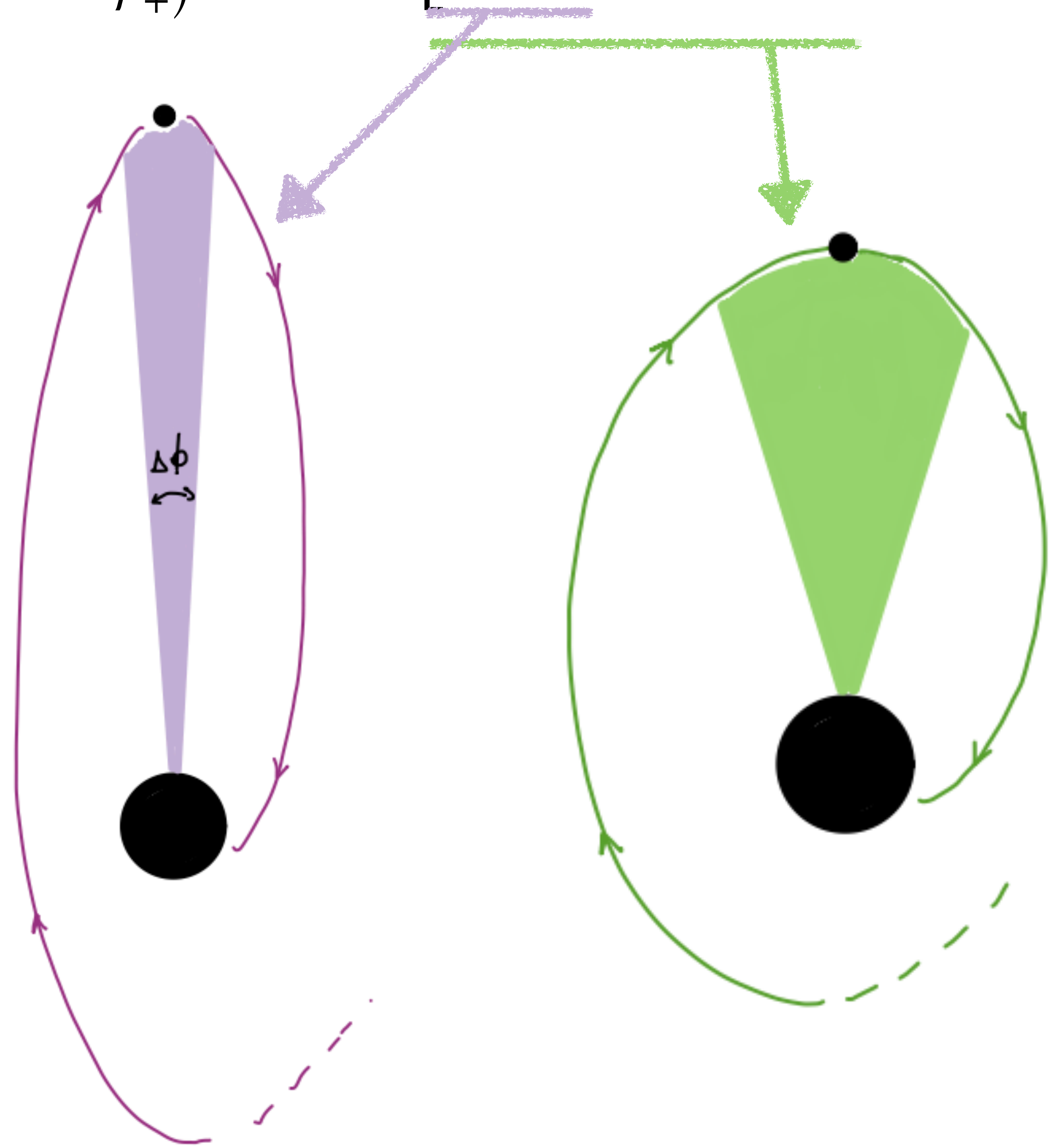

Where do we go from here?

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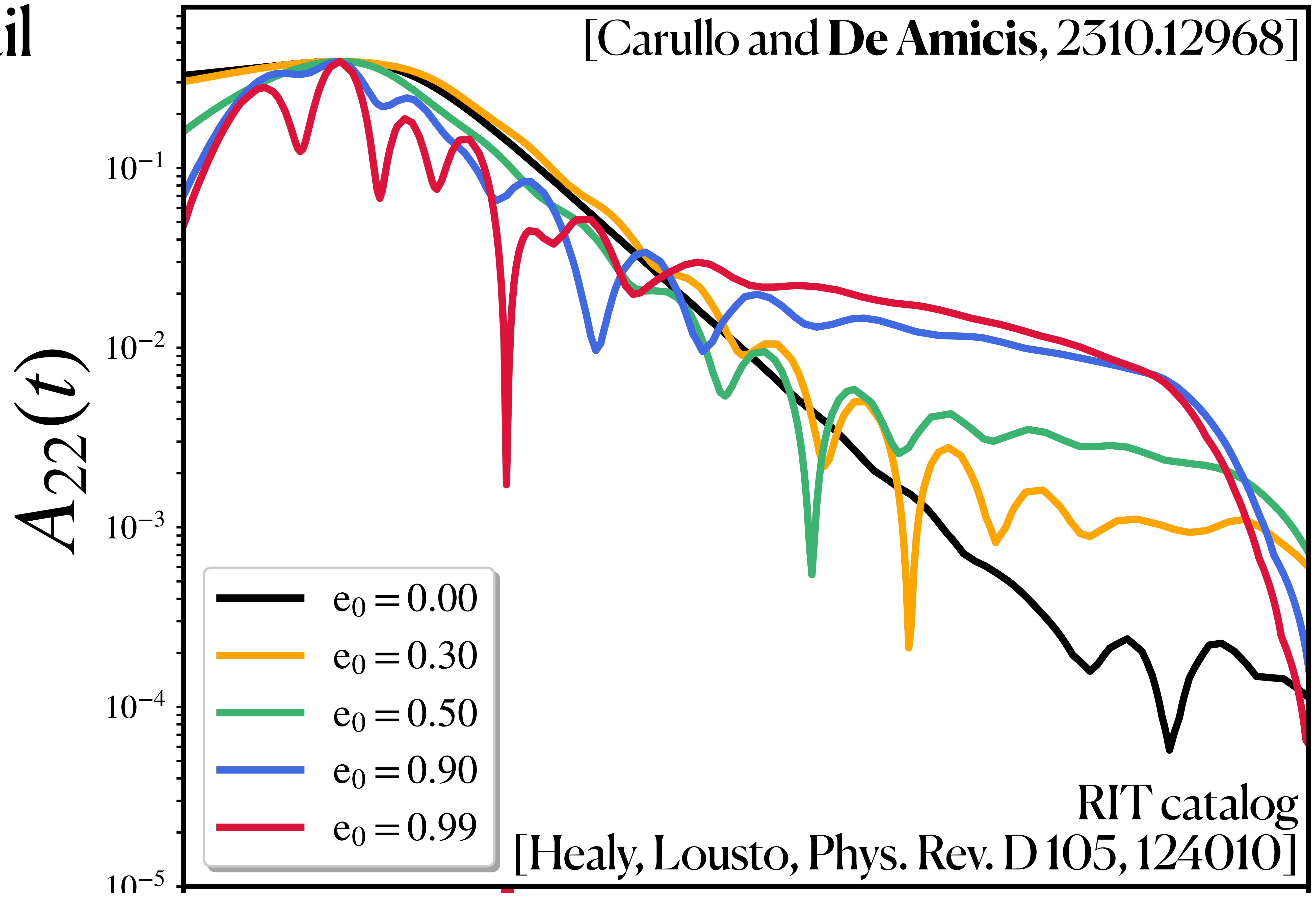
What was wrong with previous analysis

- RIT solves for Ψ^4 → suppressed tail



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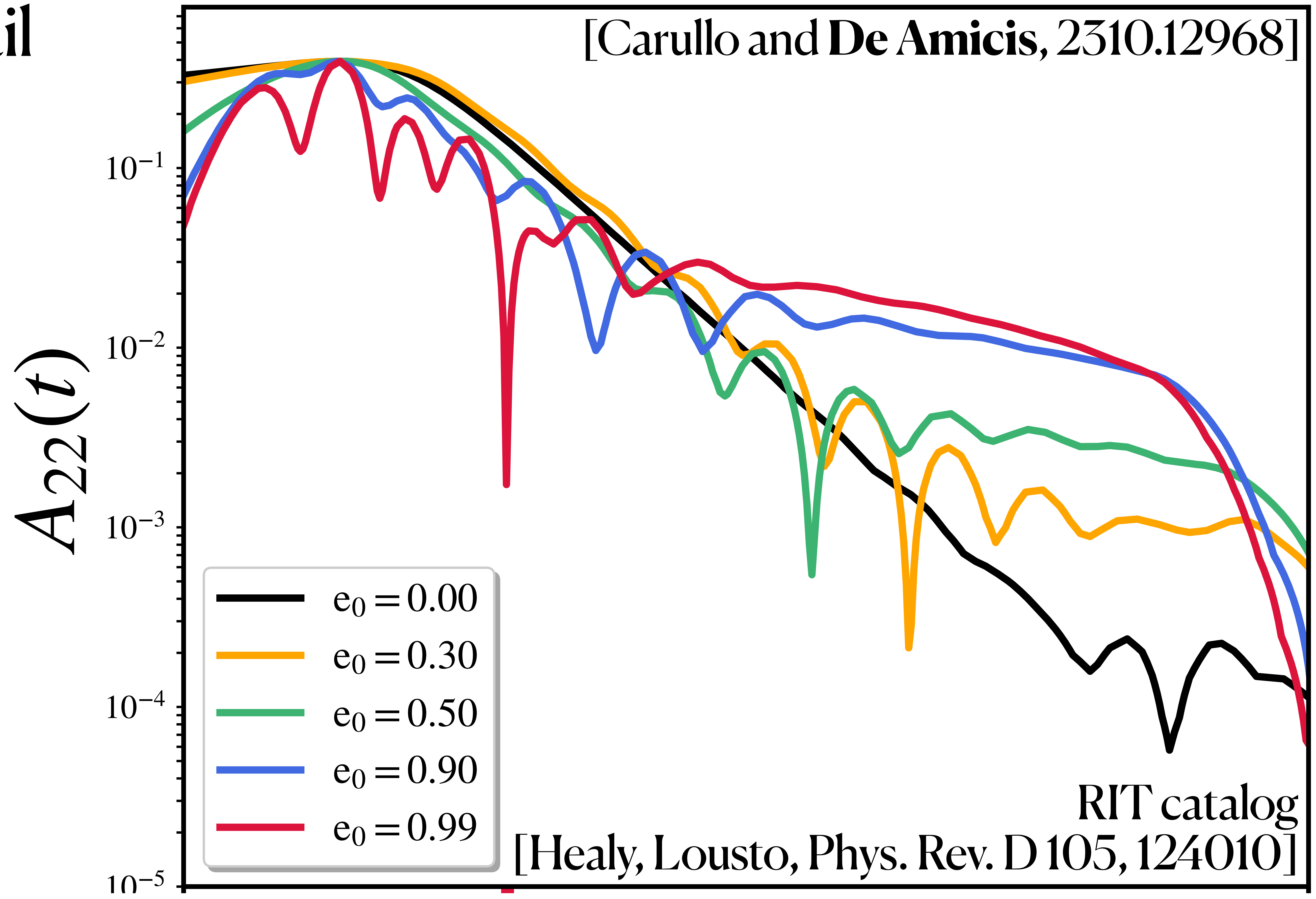
✗ Enhances noise!!



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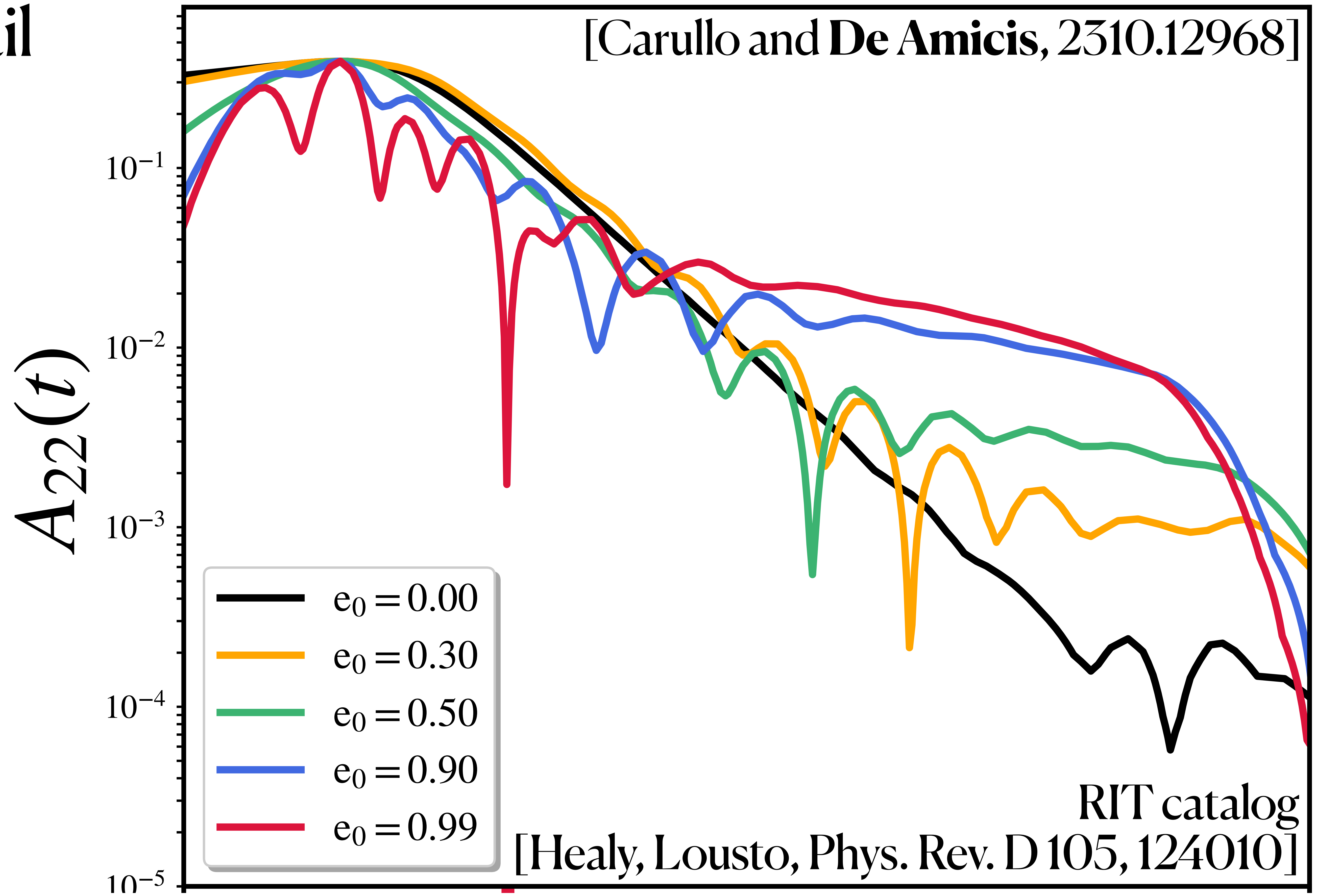

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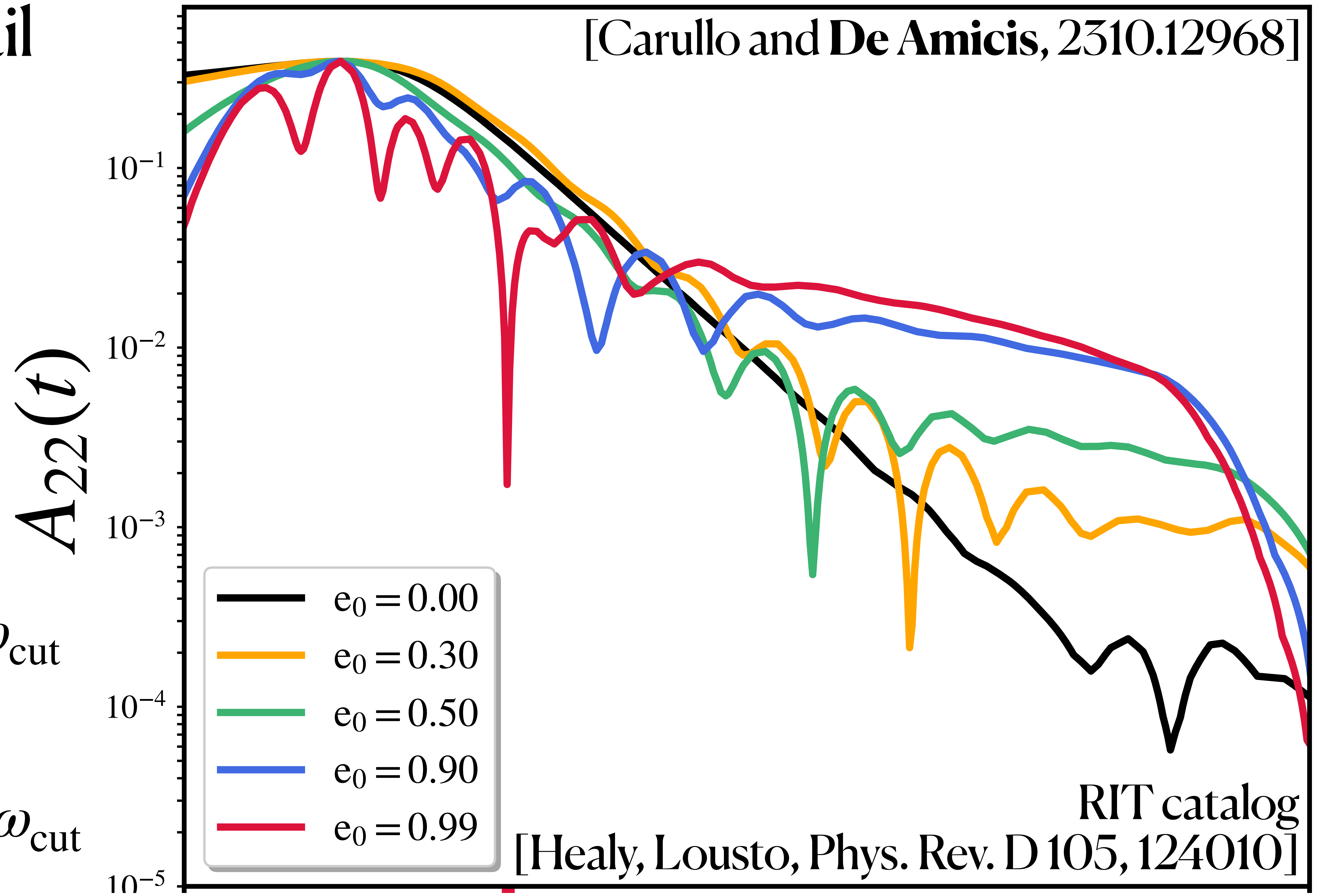
$$= \mathscr{F}^{-1}\left[\frac{i}{\omega_{\rm cut}}\tilde{H}(\omega)\right] \quad , \quad \omega \leq \omega_{\rm cut}$$

✕ ω cutoff induce spurious tails!



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  $\lim_{r\to\infty} rh(u, r) = h(u)$ as expected for asymptotically flat spacetimes

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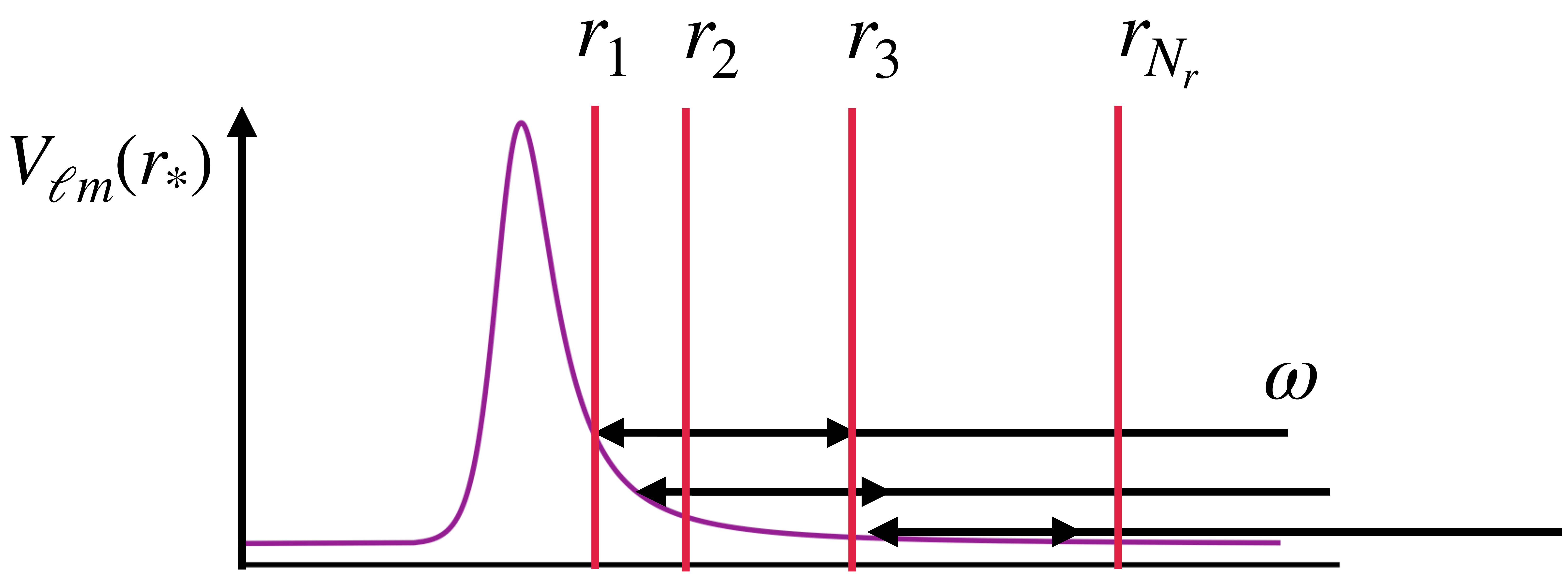
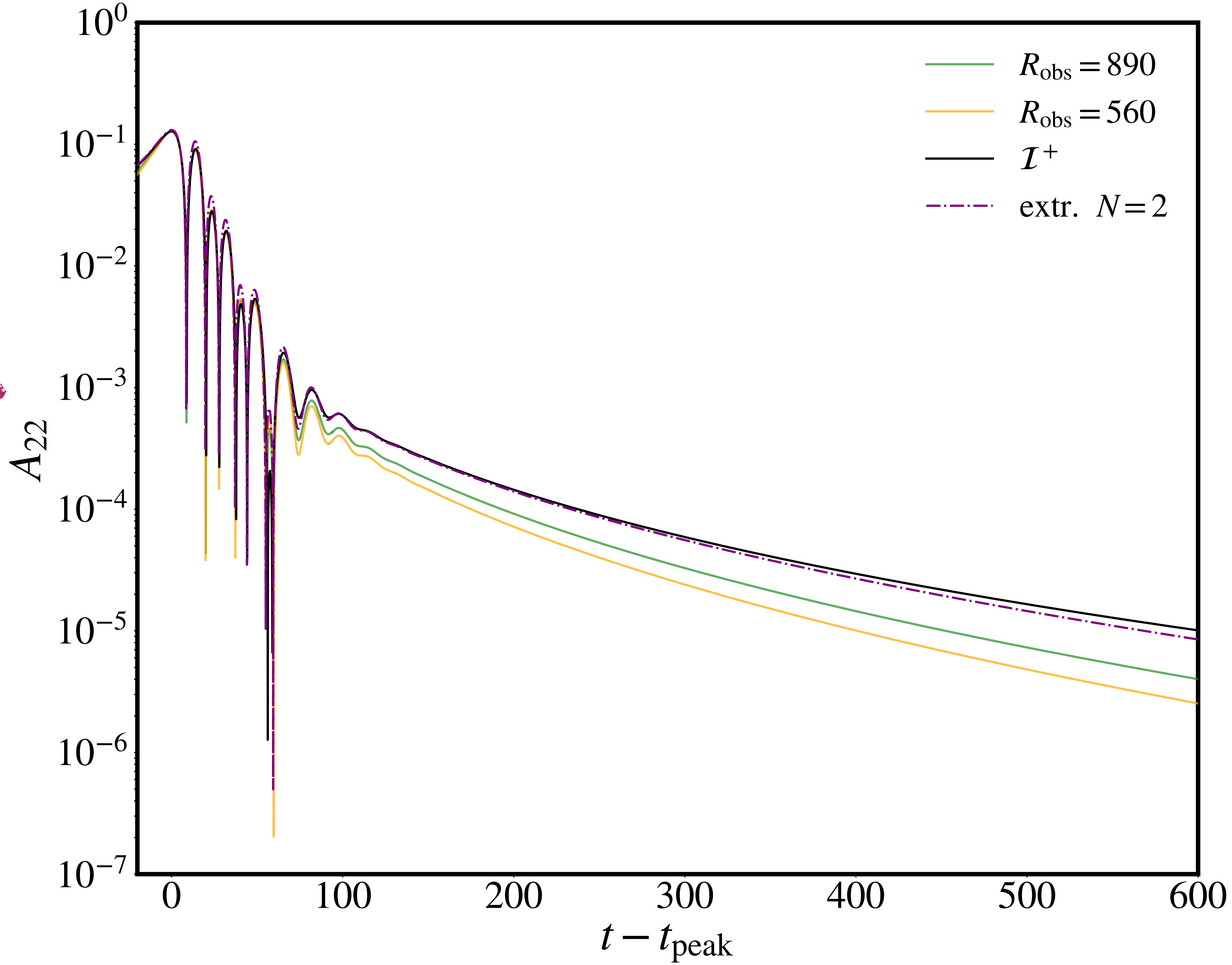
If $\{r_j\}_{N_r}$ are too close to the BH $\longrightarrow$ we are cutting tail contribution!



What was wrong with our analysis

- Large extrapolation radii needed

Test on perturbative results



Extracting tails in full numerical relativity

- SXS solves for $h(t, r)$ $\longrightarrow$ $\boxed{\text{no integrations needed!}}$

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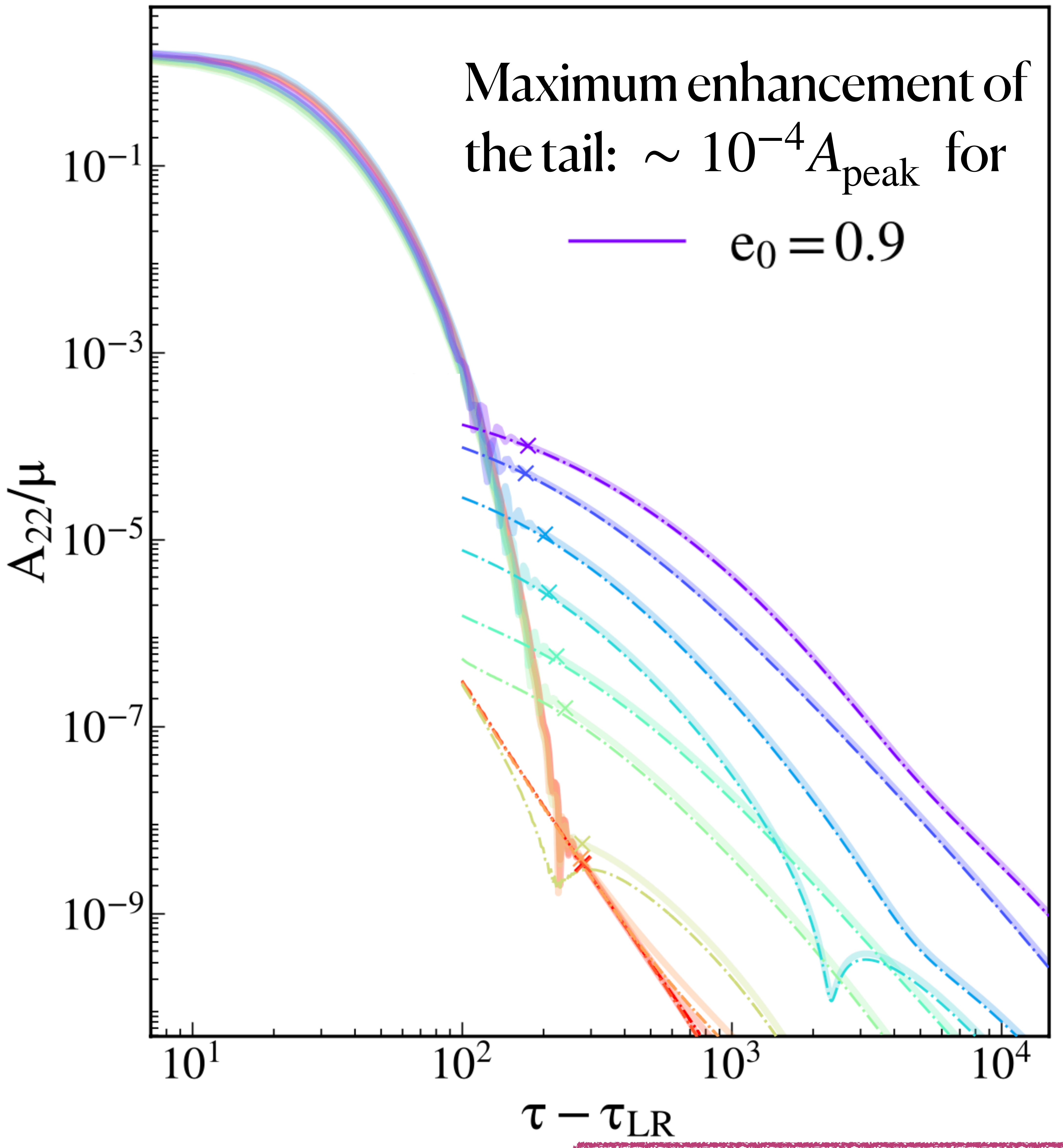
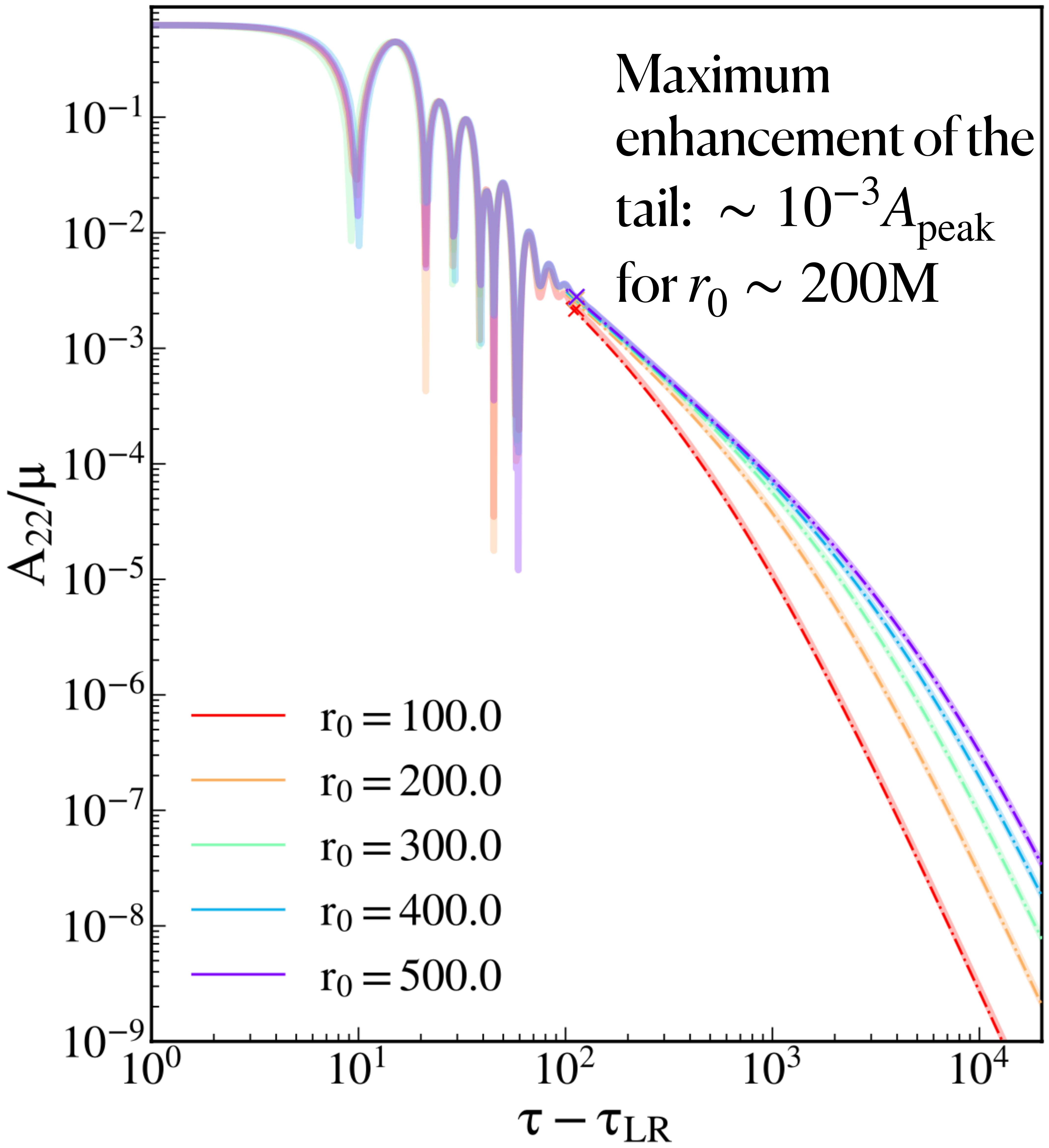
  - Maximum enhancement
  - Comparision with perturbative results

Eccentric orbits vs radial infalls (PT results)

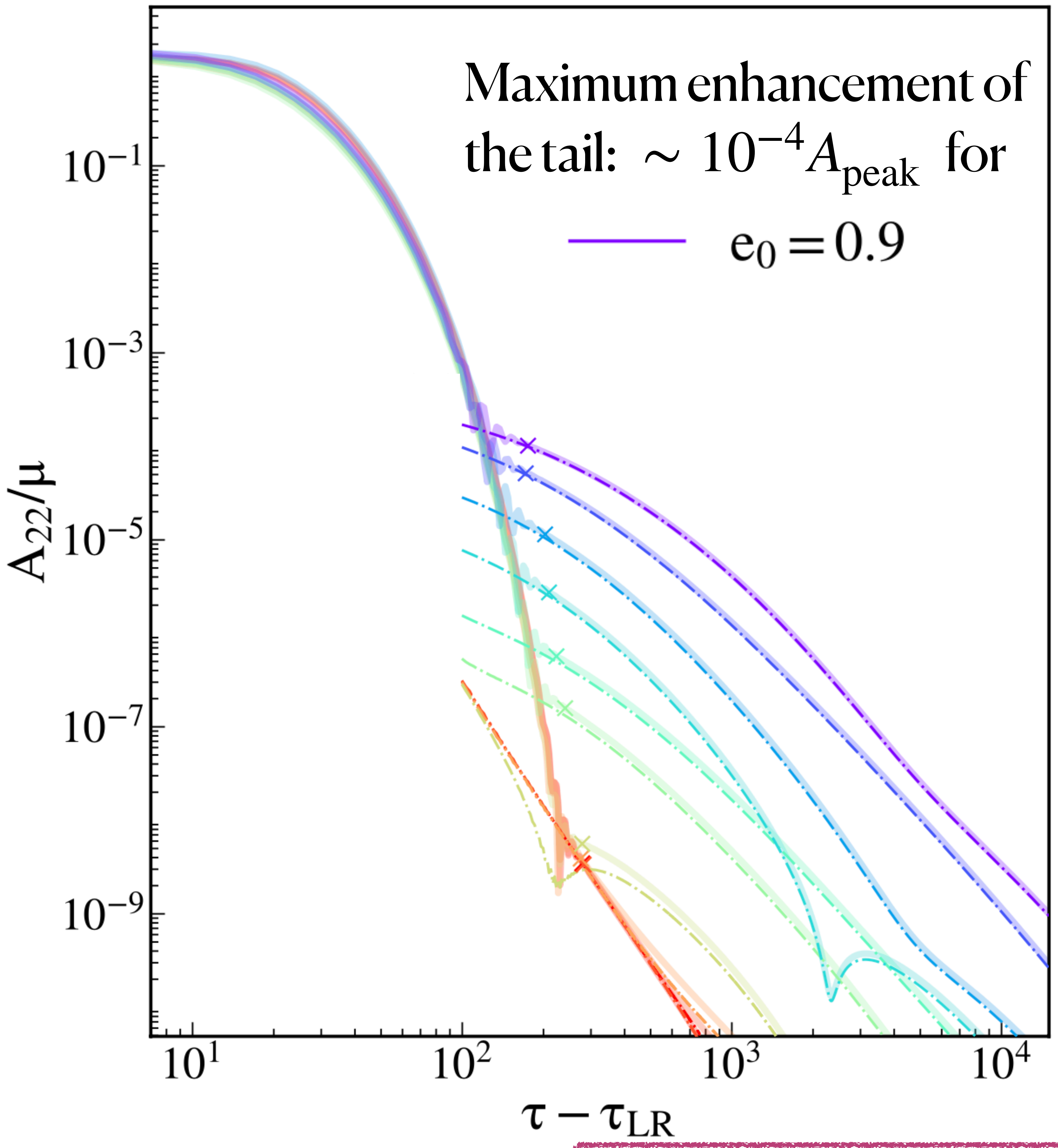


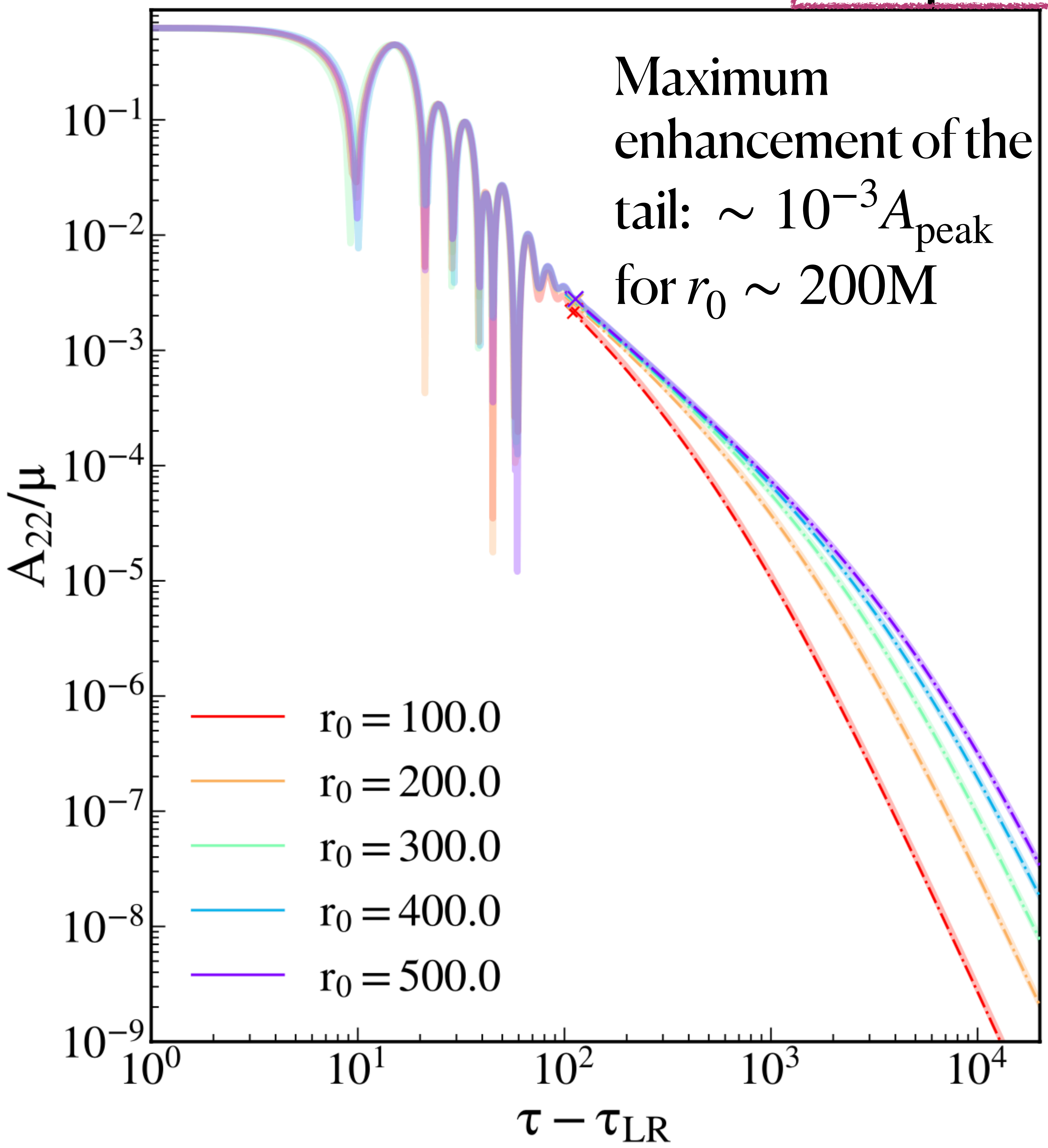


Tail emission maximied for radial infalls from $r_0 \sim 200$M

Eccentric orbits vs radial infalls (PT results)

No spin!





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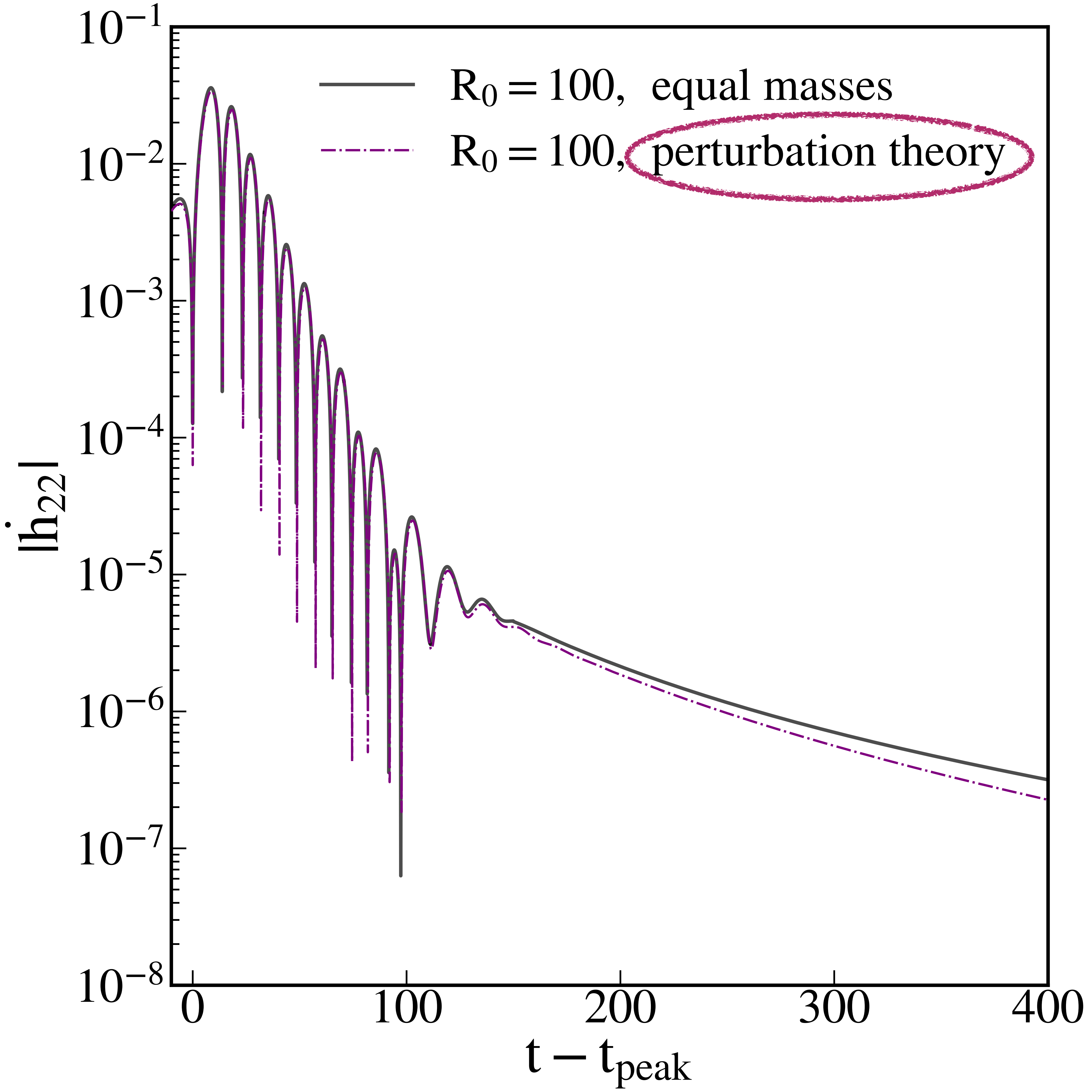
  $\longrightarrow$ Head-on collisions with different initial separations $R_0 \geq 100$

Extracting tails in full numerical relativity

- Study head-on collisions with different initial separations $R_0 \geq 100$ ✓
- Use large extraction radii for the extrapolation ✓
- Boundary of integration can contaminate the late-times waveform!

$\longrightarrow$ Push boundary at large r, such that is it **not in causal contact with the waveform** along the whole evolution

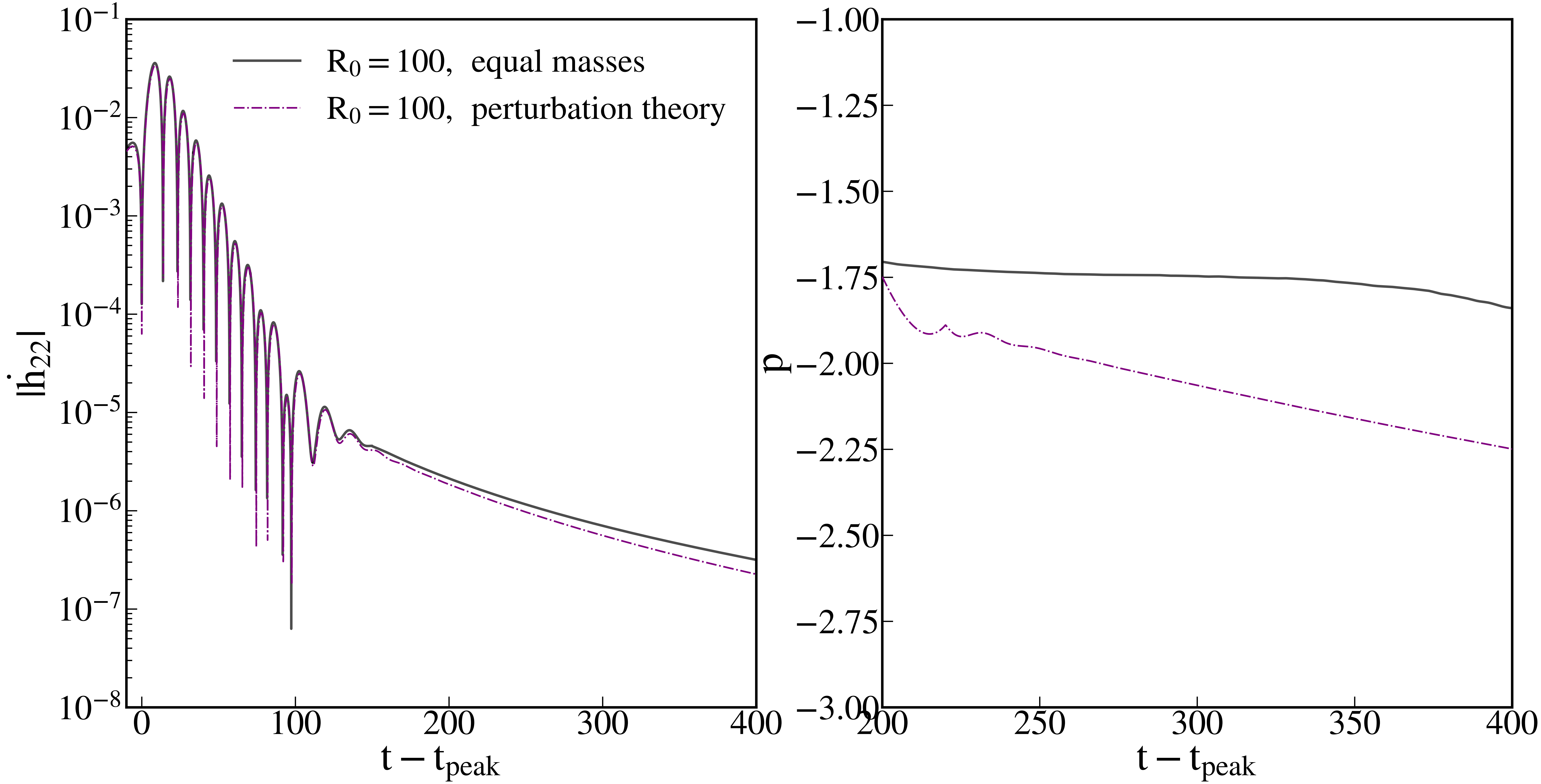
Extracting tails in full numerical relativity



- Test-particle infall initialized from the same distance R_0
- Same initial ADM energy
- Perturbative waveform rescaled by $\dfrac{1}{4\mu}$

[De Amicis et al, in preparation]

Extracting tails in full numerical relativity



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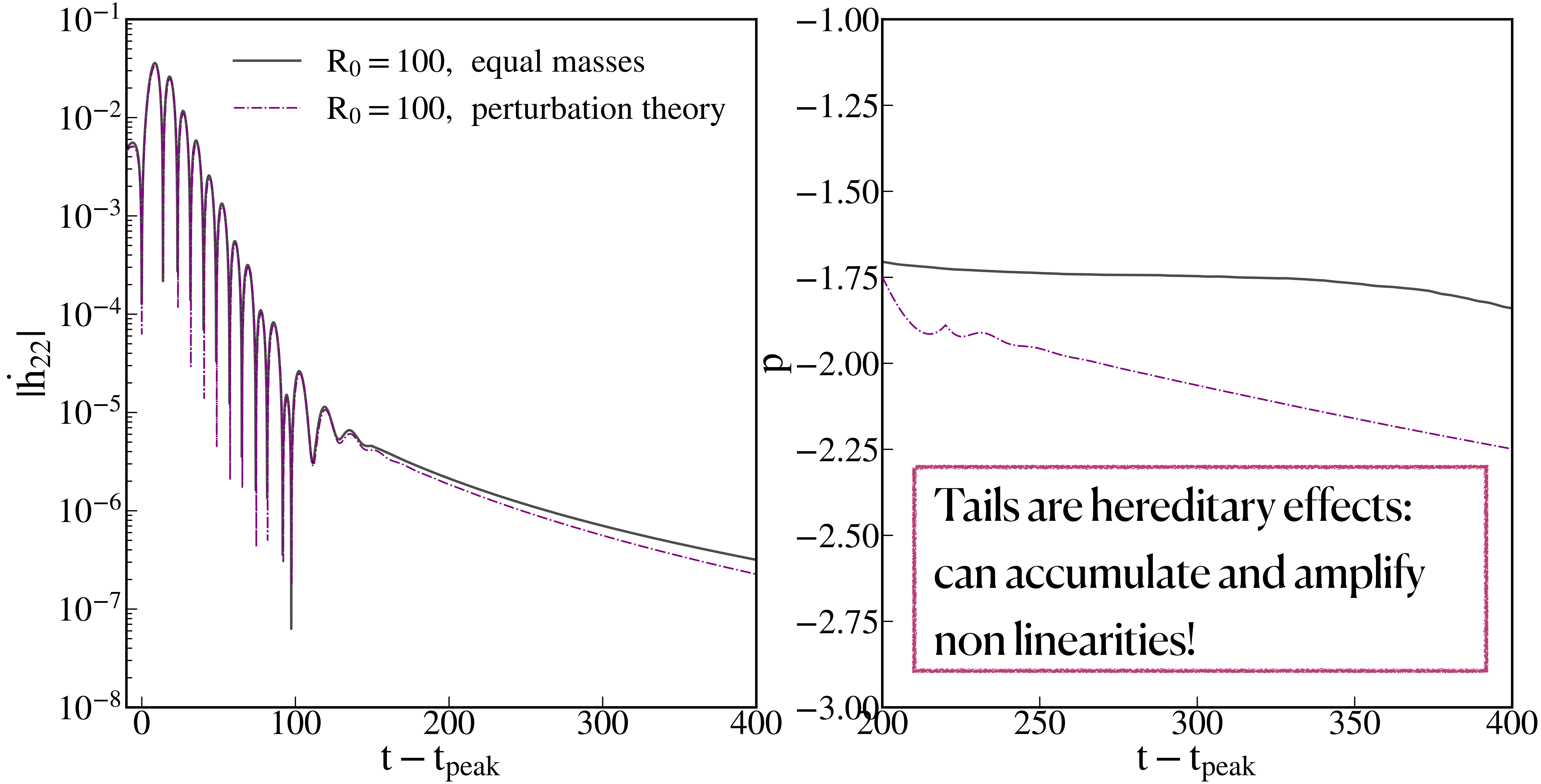


Tails are hereditary effects: can accumulate and amplify non linearities!

[De Amicis et al, in preparation]

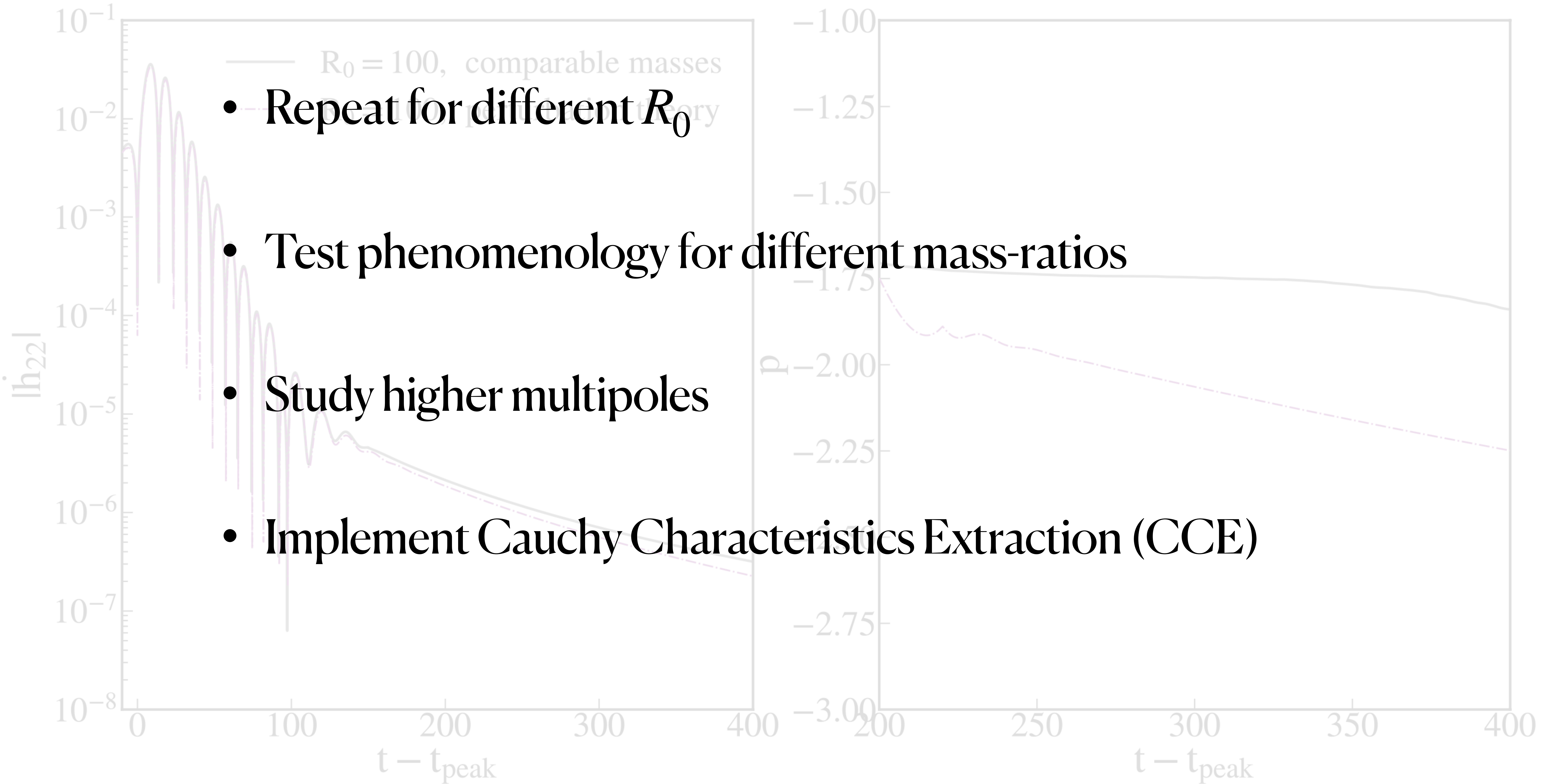
Extracting tails in full numerical relativity

Tails are hereditary effects: can accumulate and amplify non linearities!

- $\mathcal{O}(\mu^2)$ corrections in source
- Second order tails [Okuzumi et al, 0803.0501]
  [Cardoso et al (De Amicis), 2405.12290]
- Dynamical background (3rd order)

[De Amicis et al, in preparation]

Extracting tails in full numerical relativity



- Repeat for different R_0
- Test phenomenology for different mass-ratios
- Study higher multipoles
- Implement Cauchy Characteristics Extraction (CCE)

[De Amicis et al, in preparation]

Conclusions

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Future directions

- Extend the model to Kerr
    - Long-range propagator in Kerr
    - Test for EMR against Teukode [Harms et al,CQG 31, 245004(2014)]
- Perturbative study of tails in dynamical spacetimes…
- …and in environments

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BH spectroscopy: "small scale" information
Tails and memories: "large scale" information