Arbitrarily long-lived linear black hole perturbations

Javier Carballo

University of Southampton

Based on **2406.06685 with Benjamin Withers**

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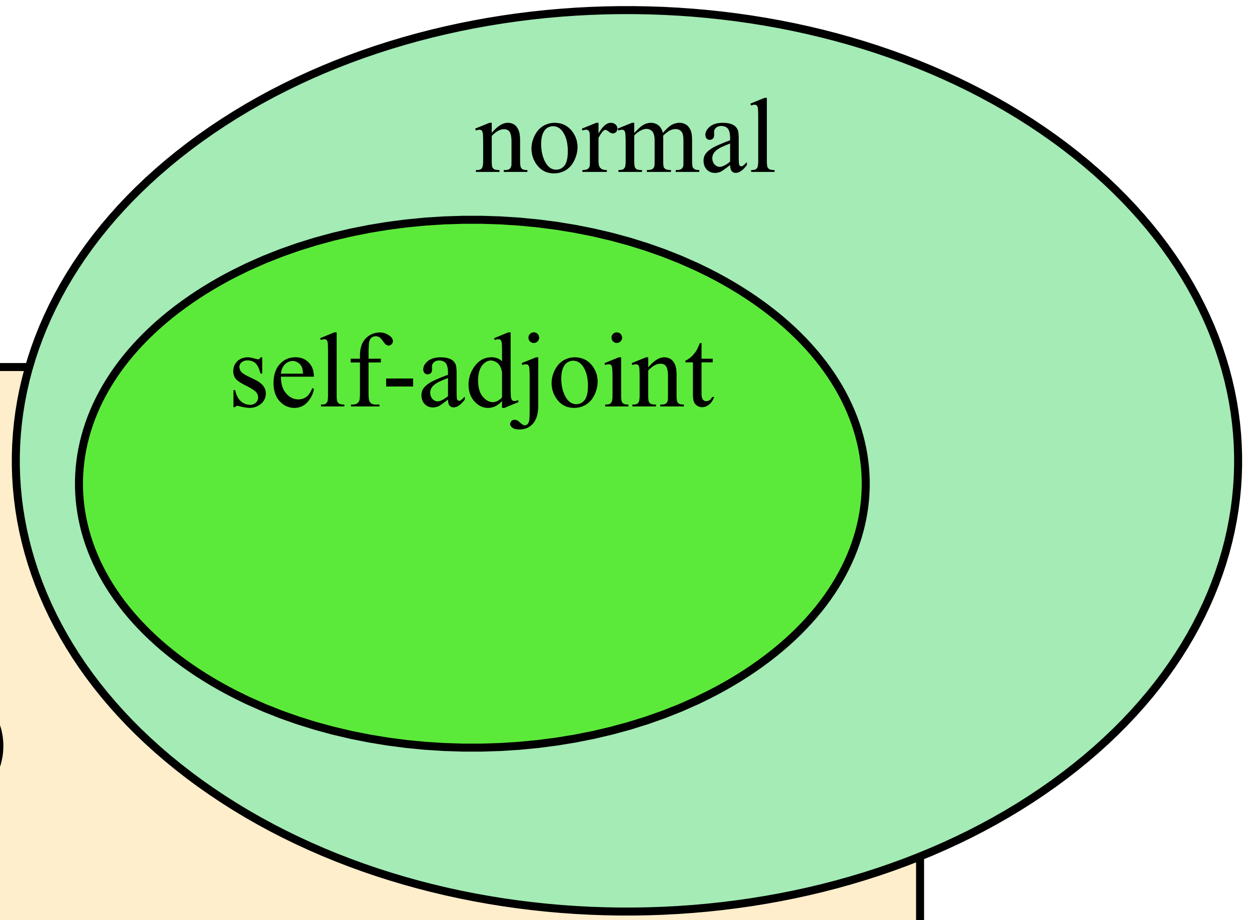
Motivation

- **Eigenvalues:** successful tool of applied mathematics (QM, PDEs, resonances, …)

⟶ **fail** to capture important effects in dynamics of systems governed by **non-normal operators**



Normal operator $\mathscr{A}$

- Complete orthonormal set of eigenfunctions (unitarily diagonalisable)
- $\left[\mathscr{A}, \mathscr{A}^{\dagger}\right] = 0$ where $\mathscr{A}^{\dagger}$ is defined by $\left\langle \xi_1, \mathscr{A}\xi_2 \right\rangle = \left\langle \mathscr{A}^{\dagger}\xi_1, \xi_2 \right\rangle$

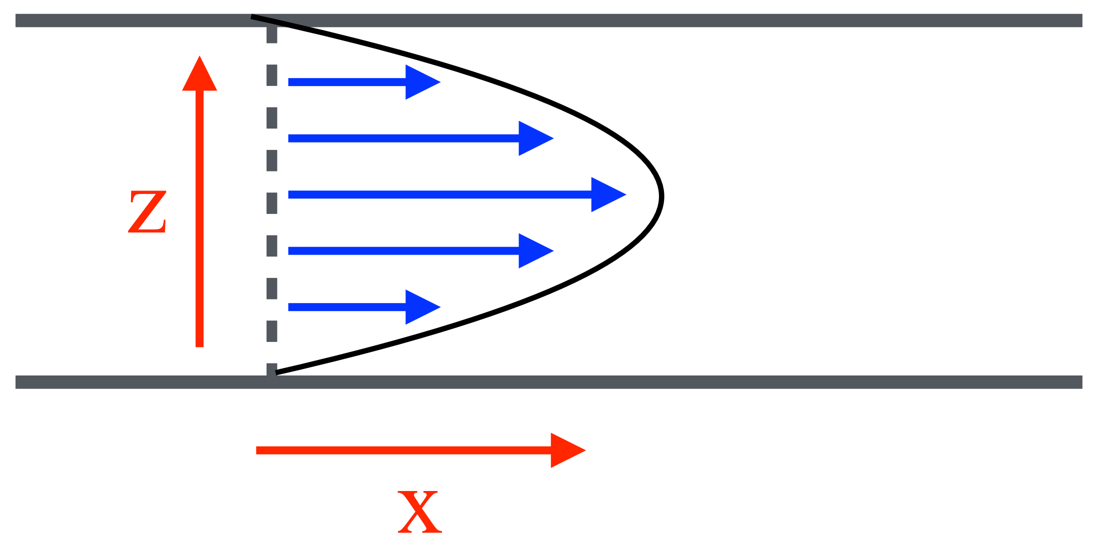
⟶ self-adjoint (Hermitian) operators $\mathscr{A} = \mathscr{A}^{\dagger}$ are a special case of normal operators

Motivation

- **Orr-Sommerfeld operator:** historical example where **eigenvalues failed**

⟶ linearised Navier-Stokes equations for plane Poiseuille flow

⟶ non-normal with respect to energy inner product

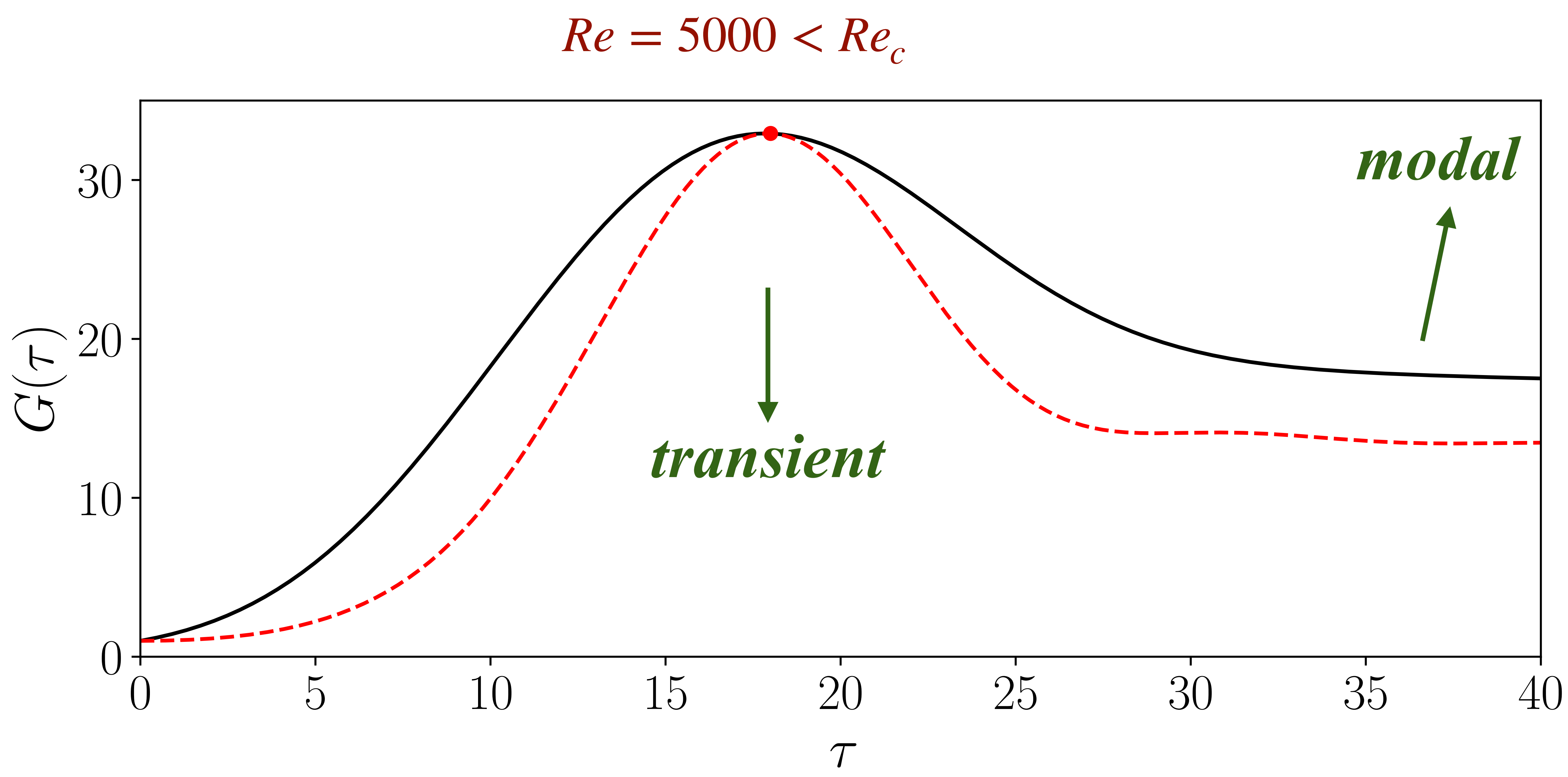


- **Eigenvalue analysis:** unstable mode at Reynolds number $Re = 5772.22$

⟶ but experiments show transition to turbulence at much lower Re

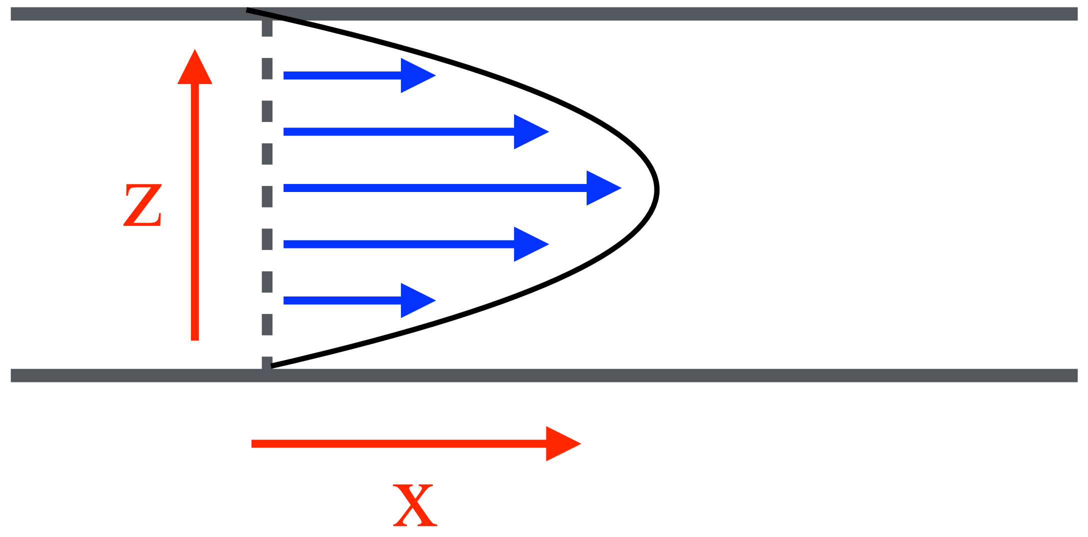
- **Non-modal study** revealed **transient growth** in energy of linear perturbations

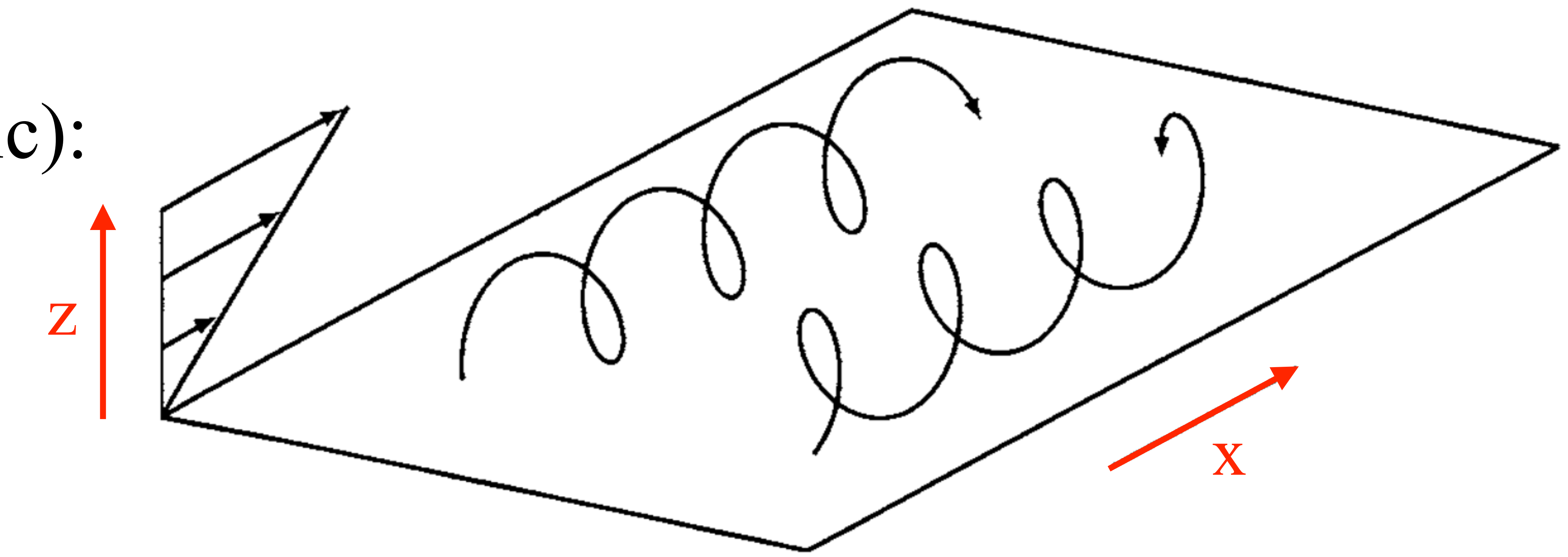
Motivation



- **Maximum** possible **energy** at each τ
- **Optimal perturbation** reaching max energy at $\tau_* = 18$

- **Initial configuration** of optimal perturbation (schematic):





[Trefethen, Embree '05]

Motivation

- $\mathscr{H}$ governing linear black hole perturbation theory is non-normal

⟶ shown to have consequences on spectral stability of QNM frequencies [Jaramillo, Macedo, Sheikh '21]

- Previous discussions of non-modal transient dynamics in gravitational systems

[Jaramillo '22] [Boyanov, Destounis, Macedo, Cardoso, Jaramillo '22]

⟶ no transient growth

This work

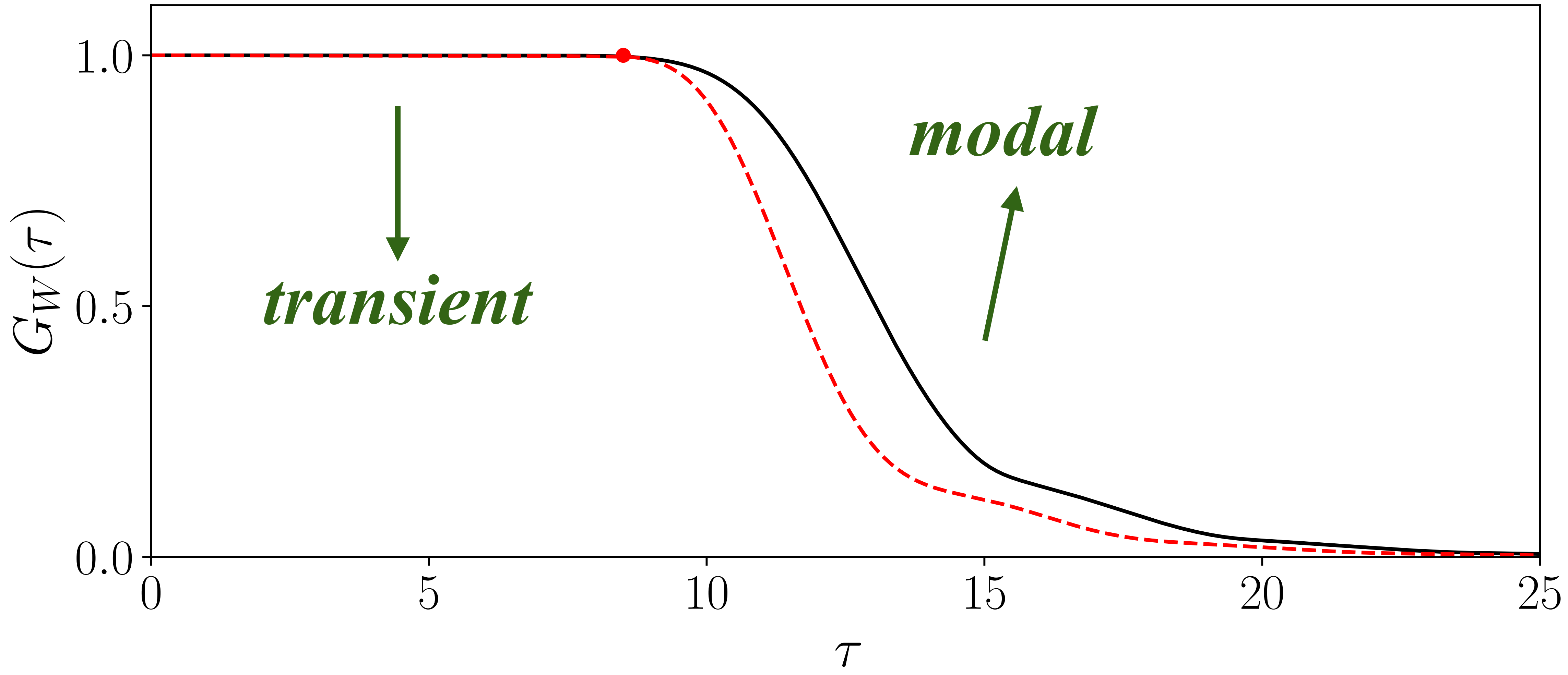
- We ask the **question:**

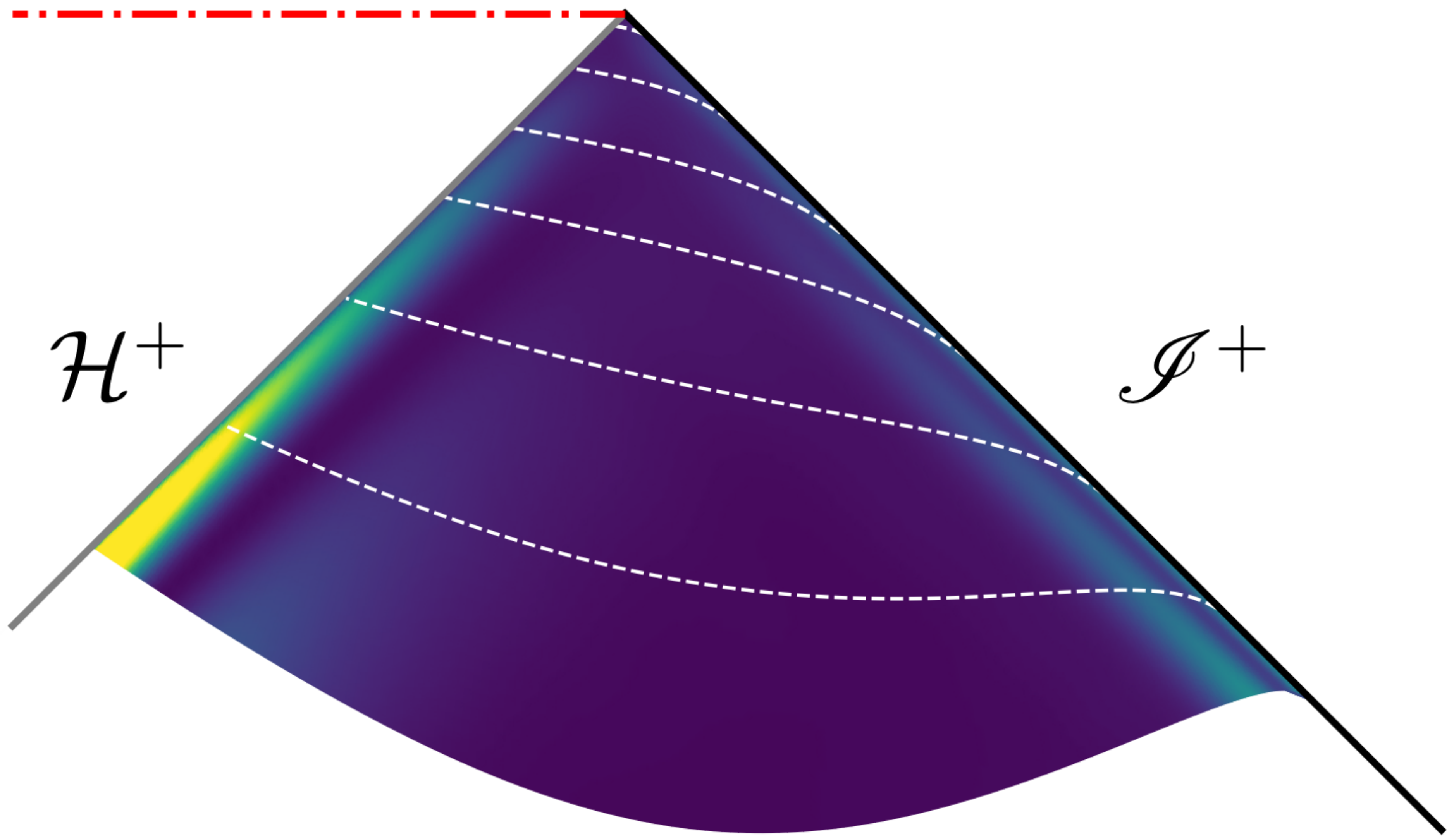
⟶ what is the **maximum response** that can be developed in **linear black hole perturbations**?

- Initiate a **systematic non-modal analysis** in gravitational systems

- We **find**:

arbitrarily long-lived sums of short-lived QNMs





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Tools and techniques

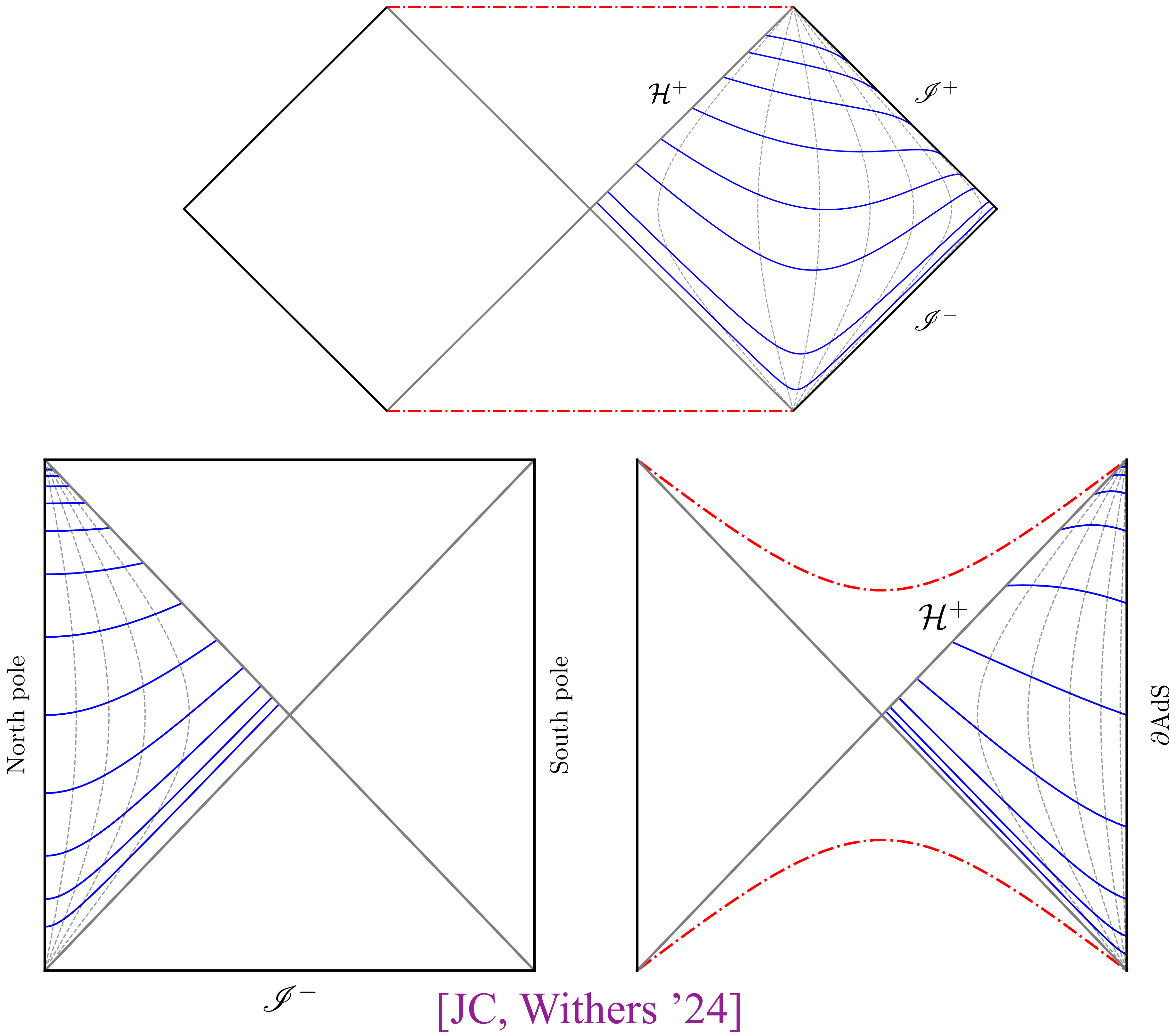
- Scalar perturbations $\left(\Box - m^2\right)\Phi = 0$ on $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{d-1}^2$

- **Hyperboloidal slices** Σ_τ

$$\begin{cases} t = \tau - h(z) \\ r = R(z) \end{cases}$$

$z = 0$: NP, ∂AdS, $\mathscr{I}^+$

$z = 1$: $\mathscr{H}^+$



[JC, Withers '24]

⟶ track energy leaving region

⟶ regular eigenfunctions of $\mathscr{H}$ are QNMs

$$i\partial_\tau \xi = \mathscr{H}\xi \qquad \mathscr{H} = \begin{pmatrix} 0 & i \\ \mathscr{L}_1 & \mathscr{L}_2 \end{pmatrix}$$

$$\xi = (\phi, \partial_\tau \phi)^T$$

$$\boxed{\xi_n(\tau, z) = e^{-i\omega_n \tau}\, \tilde{\xi}_n(z)}$$

Tools and techniques

- $\langle \cdot , \cdot \rangle$ defined by **energy on** Σ_τ : $\quad E(\tau) = \int_{\Sigma_\tau} T^{\mu}{}_{\tau}\, n_\mu d\Sigma_\tau \;\; \Rightarrow \;\; E[\xi] = \langle \xi, \xi \rangle$

  <u>outgoing energy flux</u> $\quad \mathscr{F}(\tau) \equiv -\partial_\tau E[\xi(\tau, z)]$

→ dS_{d+1} and $SAdS_{d+1}$: $\quad \mathscr{F}(\tau) = |\partial_\tau \phi|^2 \Big|_{z=1}$

no transient *growth*

→ Schwarzschild: $\quad \mathscr{F}(\tau) = |\partial_\tau \phi|^2 \Big|_{z=1} + |\partial_\tau \phi|^2 \Big|_{z=0}$

- $\mathscr{H}$ **non-normal** with respect to energy: $\quad \left\langle \xi_n, \xi_m \right\rangle \neq 0, \quad n \neq m$

→ perturbation from first M QNMs: $\quad \xi(\tau, z) = \sum_{n=1}^{M} c_n\, e^{-i\omega_n \tau}\, \tilde{\xi}_n(z)$

$$E[\xi] = \sum_{n=1}^{M} \sum_{m=1}^{M} c_n^* c_m e^{i(\omega_n^* - \omega_m)\tau} \left\langle \tilde{\xi}_n, \tilde{\xi}_m \right\rangle$$
$$= \sum_{n=1}^{M} |c_n|^2 e^{2\mathfrak{Im}\,\omega_n \tau} E[\tilde{\xi}_n] + \text{cross-terms}\,.$$

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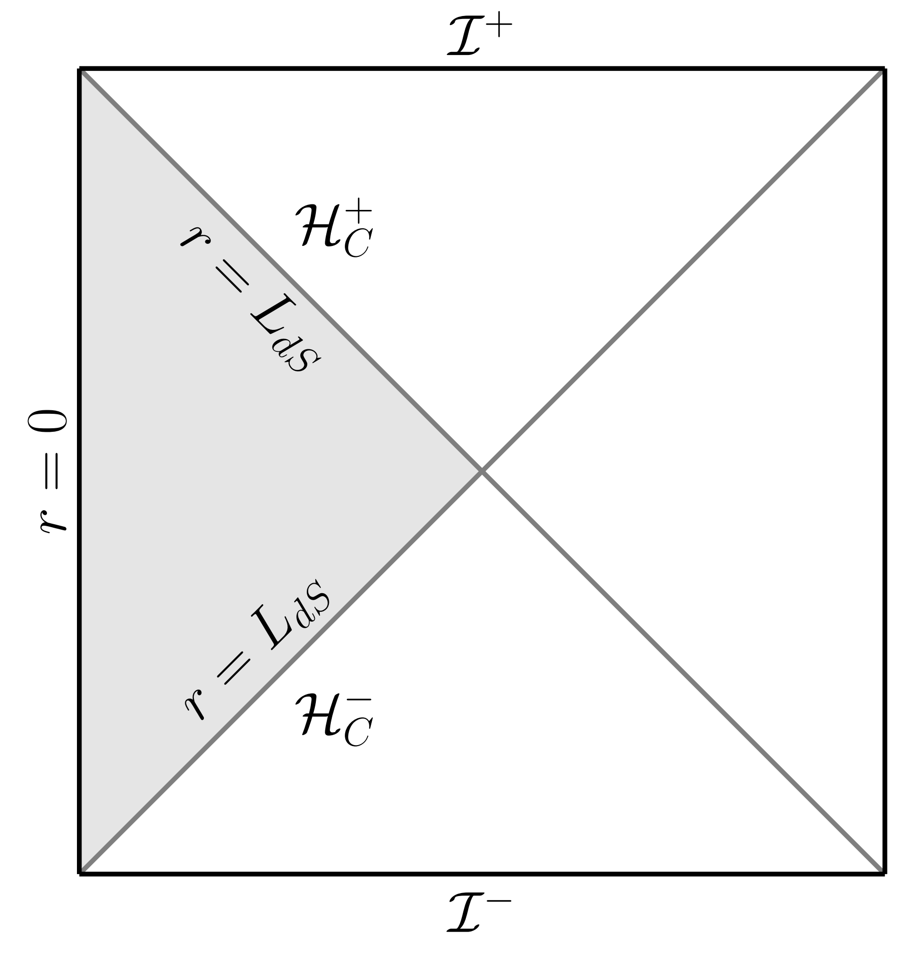
Analytical example: dS_{d+1} static patch



- QNMs are analytical: consider $d = 3,\ m^2 = 2,\ l = 0$

⟶ hypergeometric functions

- Sum of first two $n = 0,\ n = 1$: $\phi(\tau, z) = a_1\, e^{-\tau} + a_2\, e^{-2\tau}$

$$E(\tau) = \frac{1}{2}|a_1|^2 e^{-2\tau} + \frac{2}{3}(a_1^* a_2 + a_1 a_2^*)e^{-3\tau} + |a_2|^2 e^{-4\tau} \;\Rightarrow\; \frac{E(\tau)}{E(0)} = 1 - 6\frac{|a_1 + 2a_2|^2}{3|a_1|^2 + 6|a_2|^2 + 4(a_1^* a_2 + a_1 a_2^*)}\tau + O(\tau)^2$$

- Sum of first M QNMs:

$$E(\tau) = 1 - (1 - e^{-\tau})^{2M-1}\left(1 + (2M-1)e^{-\tau}\right) \;\Rightarrow\; E(\tau) = 1 - 2M\tau^{2M-1} + O(\tau)^{2M}$$

$$\lim_{M\to\infty} E(\tau) = 1 \quad \text{but} \quad \lim_{M\to\infty} \partial_z \phi\Big|_{z=1} = \infty$$

⟶ outgoing flux $\mathcal{F}(\tau)$ is peaked at $\tau = \log M$, lifetime of perturbation

- Energy can remain arbitrarily long (not infinitely)

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Optimal perturbations

- Analytical example is not the slowest energy decay. **Maximise energy** at target time τ_*

⟶ parallel techniques to Orr-Sommerfeld case [Reddy, Schmid, Henningson, '93]

- ***Energy growth* curve** $G(\tau) \equiv \sup_{\xi(0,z)} \frac{E[\xi(\tau,z)]}{E[\xi(0,z)]} = \sup_{\xi(0,z)} \frac{E[e^{-i\mathscr{H}\tau}\xi(0,z)]}{E[\xi(0,z)]} = \|e^{-i\mathscr{H}\tau}\|_E^2$ $\quad$ $e^{2\Im\mathfrak{m}\,\omega_0\tau} \leq G(\tau) \leq 1, \quad \forall\tau \geq 0$

⟶ restrict to subspace of M QNMs $\{\tilde{\xi}_n(z)\}_{n=1}^M$, W, and find orthonormal basis via Gram-Schmidt

$$\{\psi_n(z)\}_{n=1}^M \;\Rightarrow\; \langle\psi_i,\psi_j\rangle = \delta_{ij} \qquad \xi(0,z) = \sum_{n=1}^M c_n\,\tilde{\xi}_n(z) = \sum_{n=1}^M d_n\,\psi_n(z)$$

$$e^{2\Im\mathfrak{m}\,\omega_0\tau} \leq G_W(\tau) \leq G(\tau) \leq 1, \quad \forall\tau \geq 0$$

⟶ then, $G_W(\tau) = \|e^{-i\mathscr{H}_W\tau}\|_E^2 = \|e^{-iH_W\tau}\|_2^2$ where $H_W \equiv U_W D_W U_W^{-1}$

$$\left(\vec{c} = U_W^{-1}\vec{d}\;,\; D_W = \mathrm{diag}(\omega_1,\omega_2,\ldots,\omega_M)\right)$$

- **l^2-norm of a matrix**

$\|A\|_2 = \max_{\vec{x}} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} = \sqrt{\max_{\lambda\in\sigma(A^*A)}\{\lambda\}} = s_{\max}(A)$ where $s_{\max}(A)$ denotes its maximum singular value in its SVD

Optimal perturbations

- **Singular value decomposition** (SVD) of $n \times m$ matrix B

$$\underset{n\times n}{U}\,\underset{n\times m}{\Sigma}\,\underset{m\times m}{V^*} = B \quad \text{where } UU^* = VV^* = 1$$

$\longrightarrow$ thus $BV = U\Sigma$ and $B\vec{v}_{\max} = \vec{u}_{\max}\, s_{\max}(B)$, i.e. maximum response is elicited by $\vec{v}_{\max}$

- Initial data of **optimal perturbations** in W such that $E[\xi(\tau_*, z)] = G_W(\tau_*)$:

$$\xi(0,z) = \sum_{n=1}^{M} c_n \tilde{\xi}_n(z) = \sum_{n=1}^{M} d_n \psi_n(z) \quad \text{via SVD of } e^{-iH_W\tau_*}:$$

$\longrightarrow$ $\vec{d} = \vec{v}_{\max}$ of $e^{-iH_W\tau_*}$

Optimal perturbations

Slowest possible energy decay?

→ Maximise energy at each τ: *energy growth* curve
$G(\tau) = \|e^{-i\mathscr{H}\tau}\|_E^2$

→ Restrict to subspace W, spanned by M QNMs $\{\tilde{\xi}_n(z)\}_{n=1}^M$:
$G_W(\tau) = \|e^{-i\mathscr{H}_W\tau}\|_E^2$

→ Find orthonormal basis for W wrt energy: Gram-Schmidt
$\{\psi_n(z)\}_{n=1}^M \quad \Rightarrow \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$

→ In orthonormal basis:
$G_W(\tau) = \|e^{-iH_W\tau}\|_2^2$
H_W is $M \times M$ matrix

→ Maximum possible energy at τ_* given by $G_W(\tau_*)$: 2-norm of a matrix

→ Optimal perturbation with maximum possible energy at τ_*: initial data from SVD

Optimal perturbations: Schwarzschild

- Standard treatment of perturbations: **Regge-Wheeler-Zerilli equations**
- Same **energy inner product**: closely following [Jaramillo, Macedo, Sheikh ’21]
- **Numerics**: Chebyshev spectral methods in radial z coordinate

⟶ discretise $\mathscr{H}$ into $2(N+1) \times 2(N+1)$ matrix

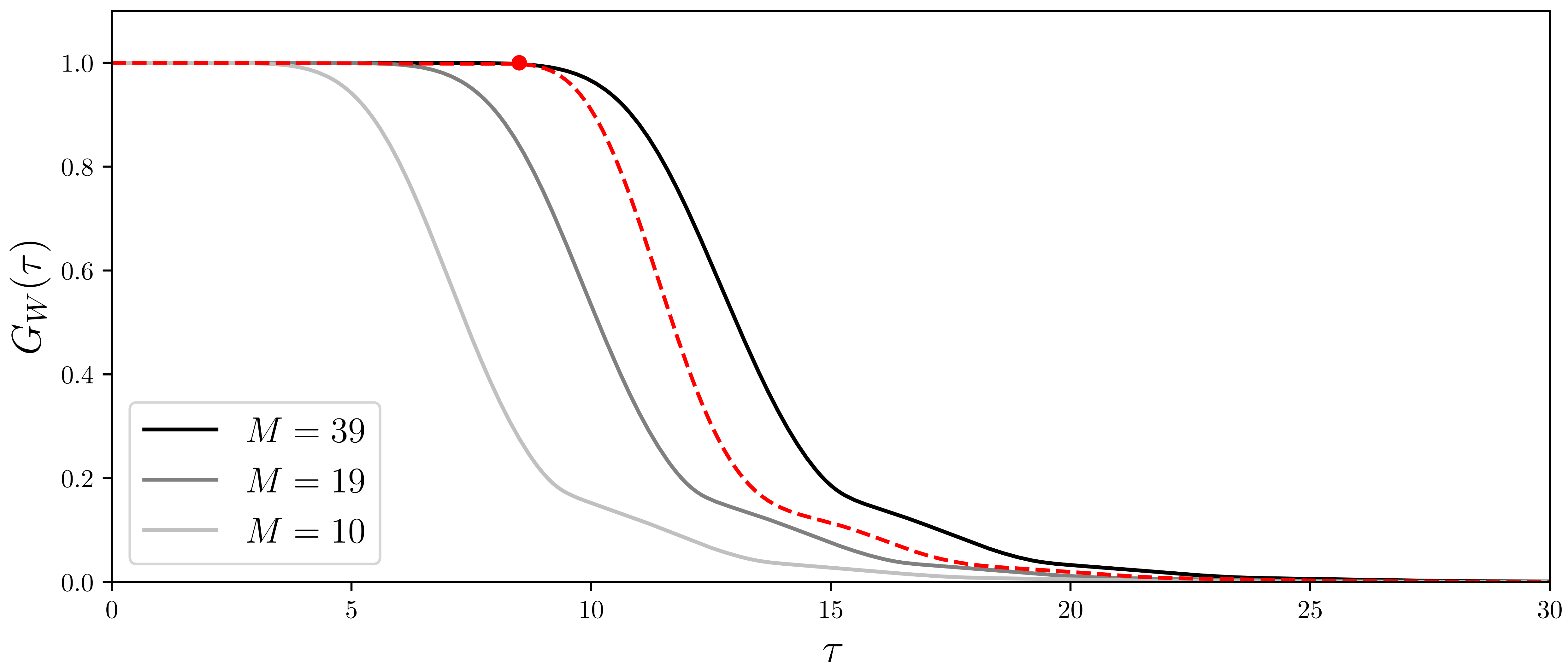
⟶ discretise energy norm $\left\langle \vec{\xi}_1, \vec{\xi}_2 \right\rangle = \vec{\xi}_1^{\,*}\, G\, \vec{\xi}_2$

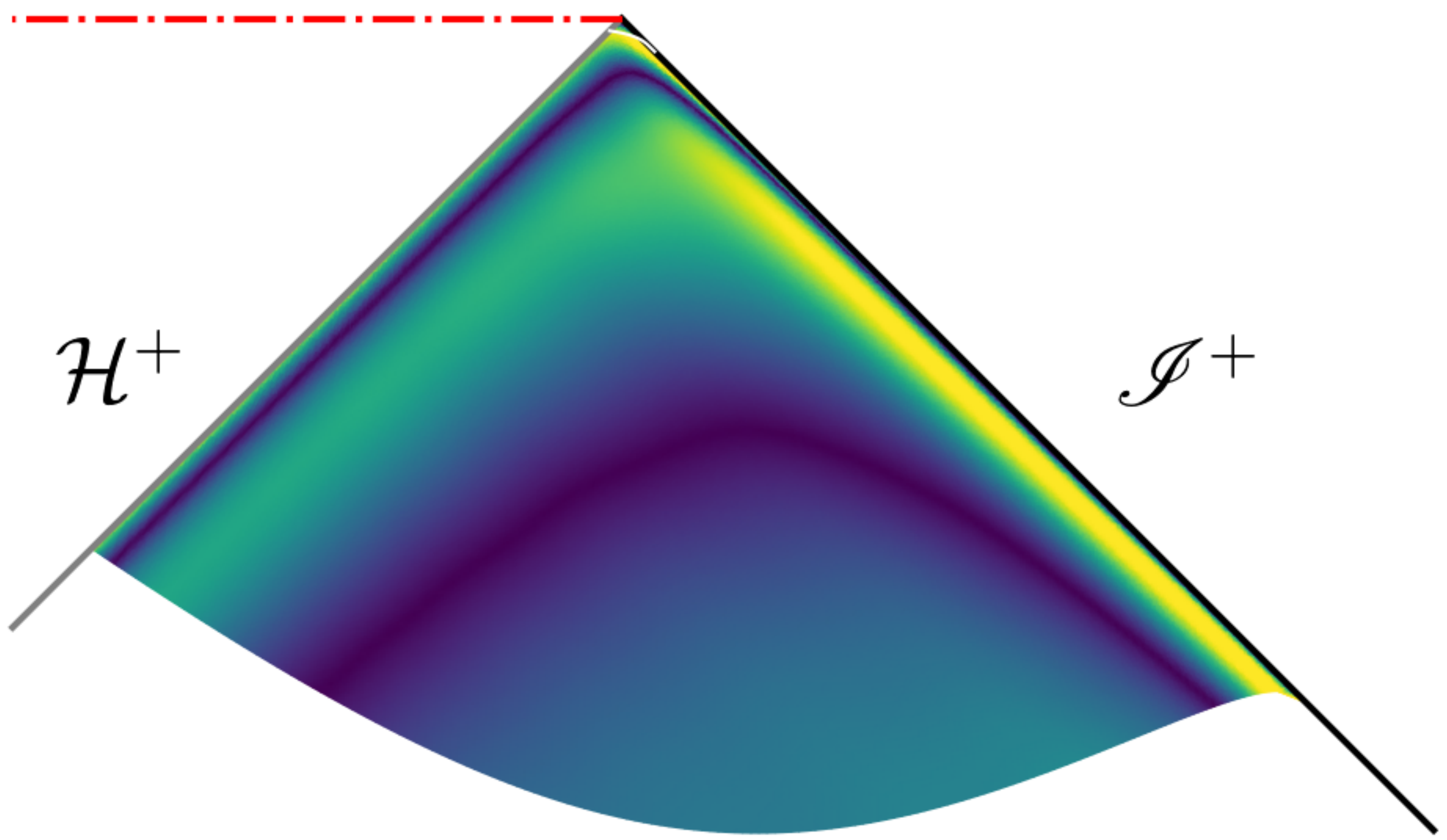
⟶ Gram-Schmidt → QR decomposition

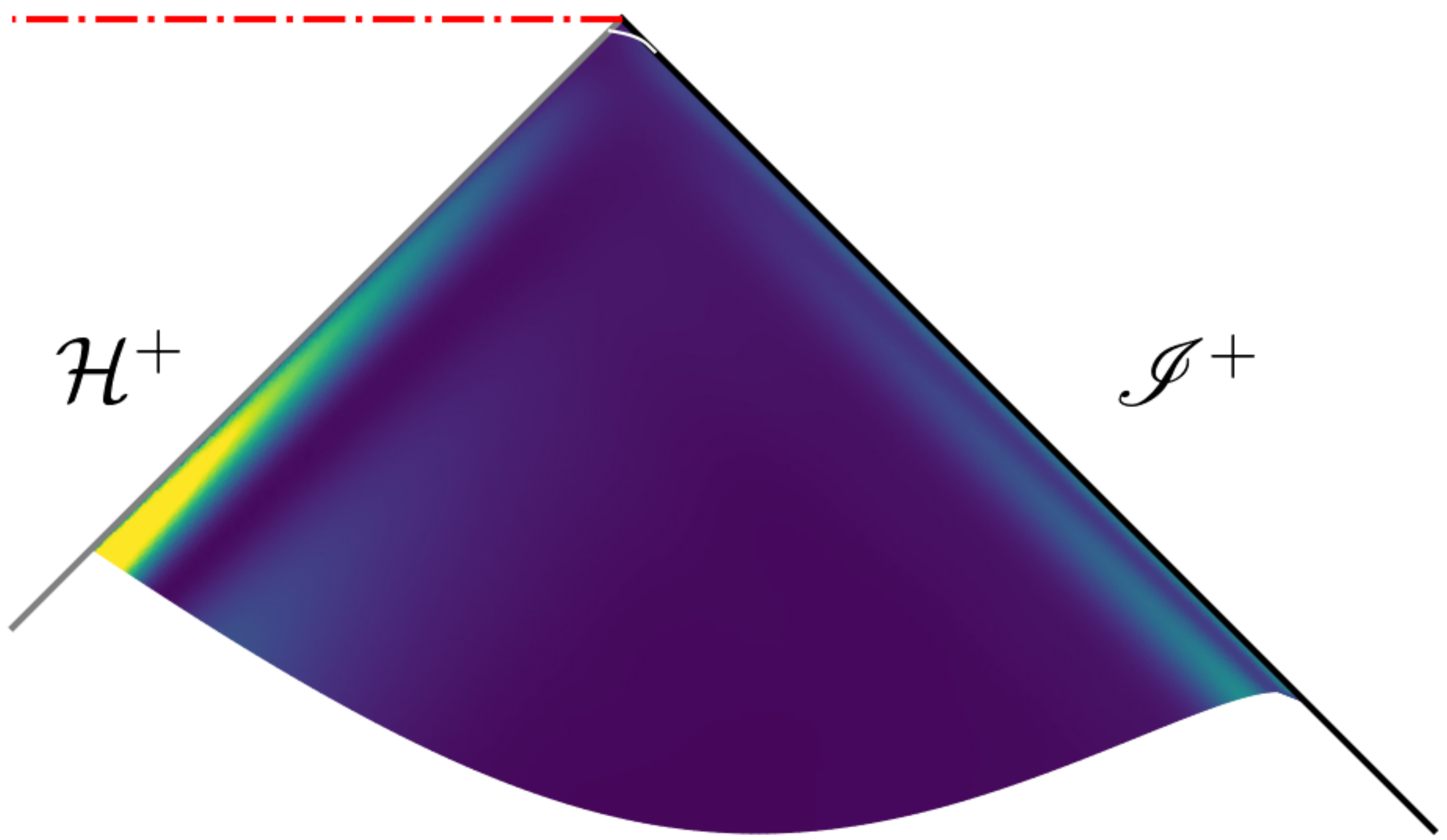
Optimal perturbations: Schwarzschild

$s = 2, l = 2$

$\tau_* = 8.5$ for $M = 39$

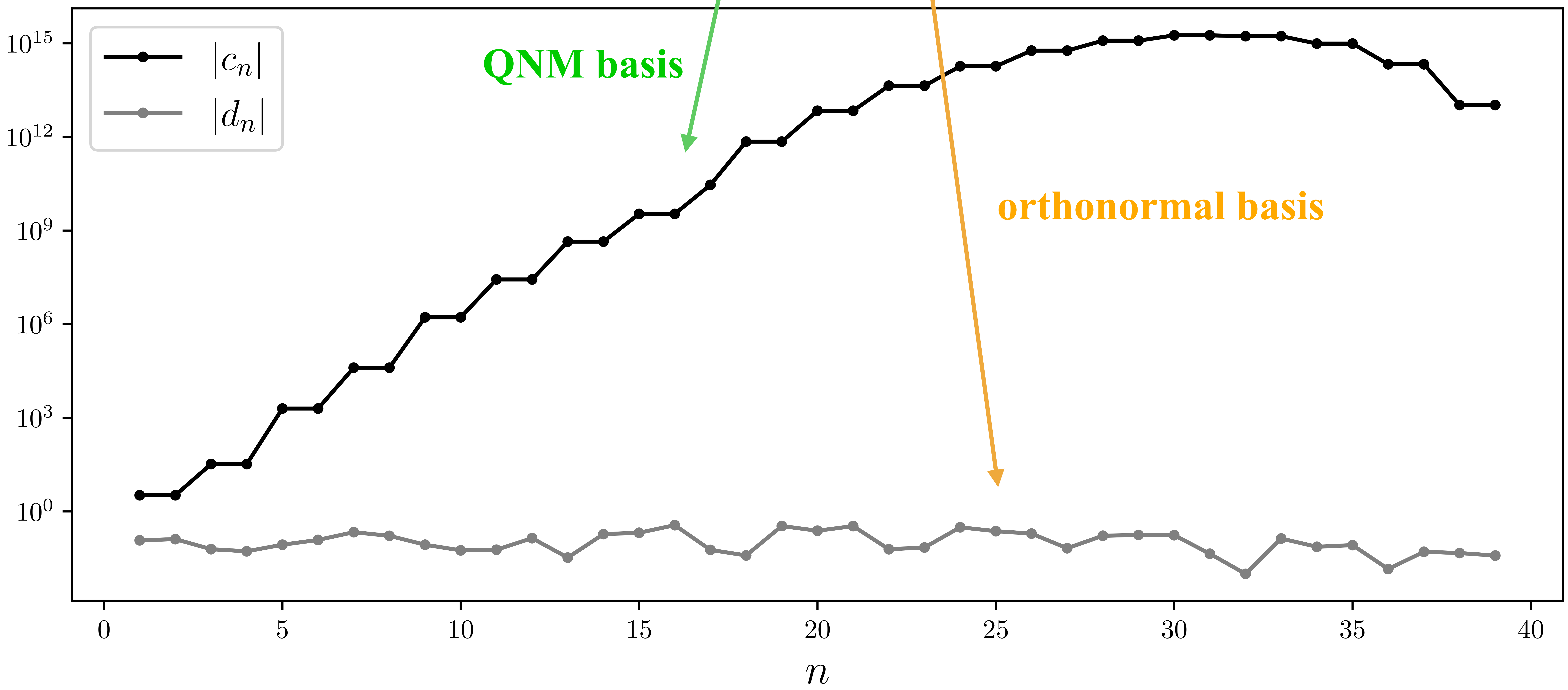






Optimal perturbations: Schwarzschild

$$\xi(0,z) = \sum_{n=1}^{39} c_n \tilde{\xi}_n(z) = \sum_{n=1}^{39} d_n \psi_n(z)$$



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Discussion

- Arbitrarily long-lived ($\sim \log M$) sums of decaying QNMs. Localised at **horizon**.

- Lifetime arbitrarily large but **finite**. Ultimate decay via longest-lived QNM

- **Universal** in Schw.-AdS, dS static patch, flat Schw.

  $\longrightarrow$ observable interest in domains where horizons are present: astrophysics, analogue black holes, condensed matter systems through AdS/CFT, etc.

- **Fine-tuning?**

- **Backreaction**

Thank you!

A. Relation to pseudospectra

- $\sigma_\epsilon(\mathscr{H}) = \{\omega \in \mathbb{C} : \|R(\omega;\mathscr{H})\| \geq \epsilon^{-1}\}$
- $\sigma_\epsilon(\mathscr{H}) = \{\omega \in \mathbb{C} : \omega \in \sigma(\mathscr{H} + \delta\mathscr{H}), \|\delta\mathscr{H}\| \leq \epsilon\}$

Theorem 15.4 of [Trefethen, Embree '05]**:**

$\mathscr{K}^2(\mathscr{H}) \leq \sup_{\tau \geq 0} G(\tau)$ where Kreiss constant is defined: $\mathscr{K}(\mathscr{H}) = \sup_{\mathfrak{Im}\,\omega > 0} (\mathfrak{Im}\,\omega)\|R(\omega;\mathscr{H})\|_E$

- no growth bound on $G(\tau)$, $G(\tau) \leq 1$, translates into $\|R(\omega;\mathscr{H})\|_E \leq \dfrac{1}{\mathfrak{Im}\,\omega}$ $\forall \omega \in \mathbb{C}$ s.t. $\mathfrak{Im}\,\omega > 0$

or equivalently $\mathfrak{Im}\,\omega \leq \epsilon$ $\forall \omega \in \sigma_\epsilon(\mathscr{H})$ s.t. $\mathfrak{Im}\,\omega > 0$ $\rightarrow$ **bound on pseudospectrum contours in UHP**

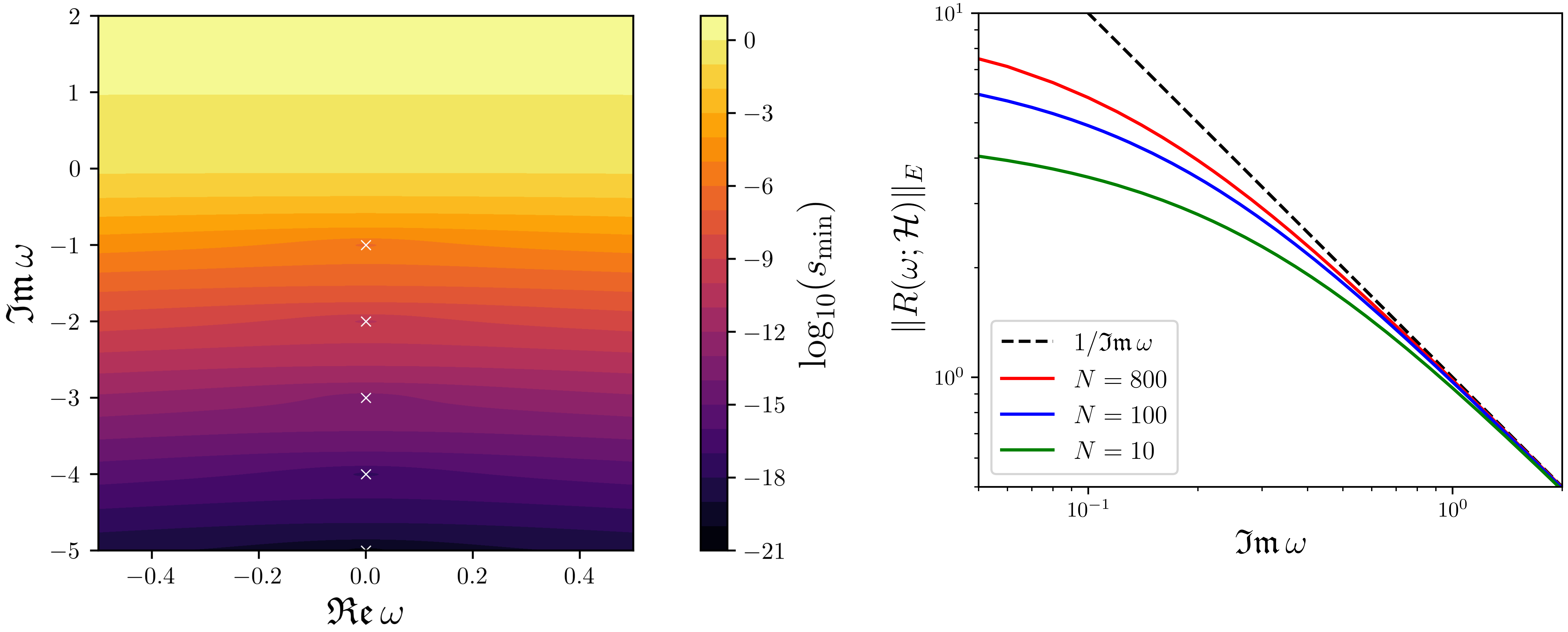
Theorem 17.4 of [Trefethen, Embree '05]**:**

$$G'(0) = 2 \sup_{\epsilon > 0} \left(\sup_{\omega \in \sigma_\epsilon(\mathscr{H})} \mathfrak{Im}\,\omega - \epsilon \right)$$

- we see $G'(0) = 0$

A. Relation to pseudospectra

dS_{d+1}: $d = 3,\ m^2 = 2,\ l = 0$



B. Decomposition of Φ

$$\Phi(\tau, z, \Omega_{d-1}) = \sum_{l\mathbf{m}} z^{\beta}\phi_l(\tau, z)Y_{l\mathbf{m}}(\Omega_{d-1})$$

$$\Phi(\tau, z, \vec{x}) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} z^{\beta}\phi_{\vec{k}}(\tau, z)e^{i\vec{k}\cdot\vec{x}}$$

C. Hyperboloidal slicing

- dS_{d+1} static patch:

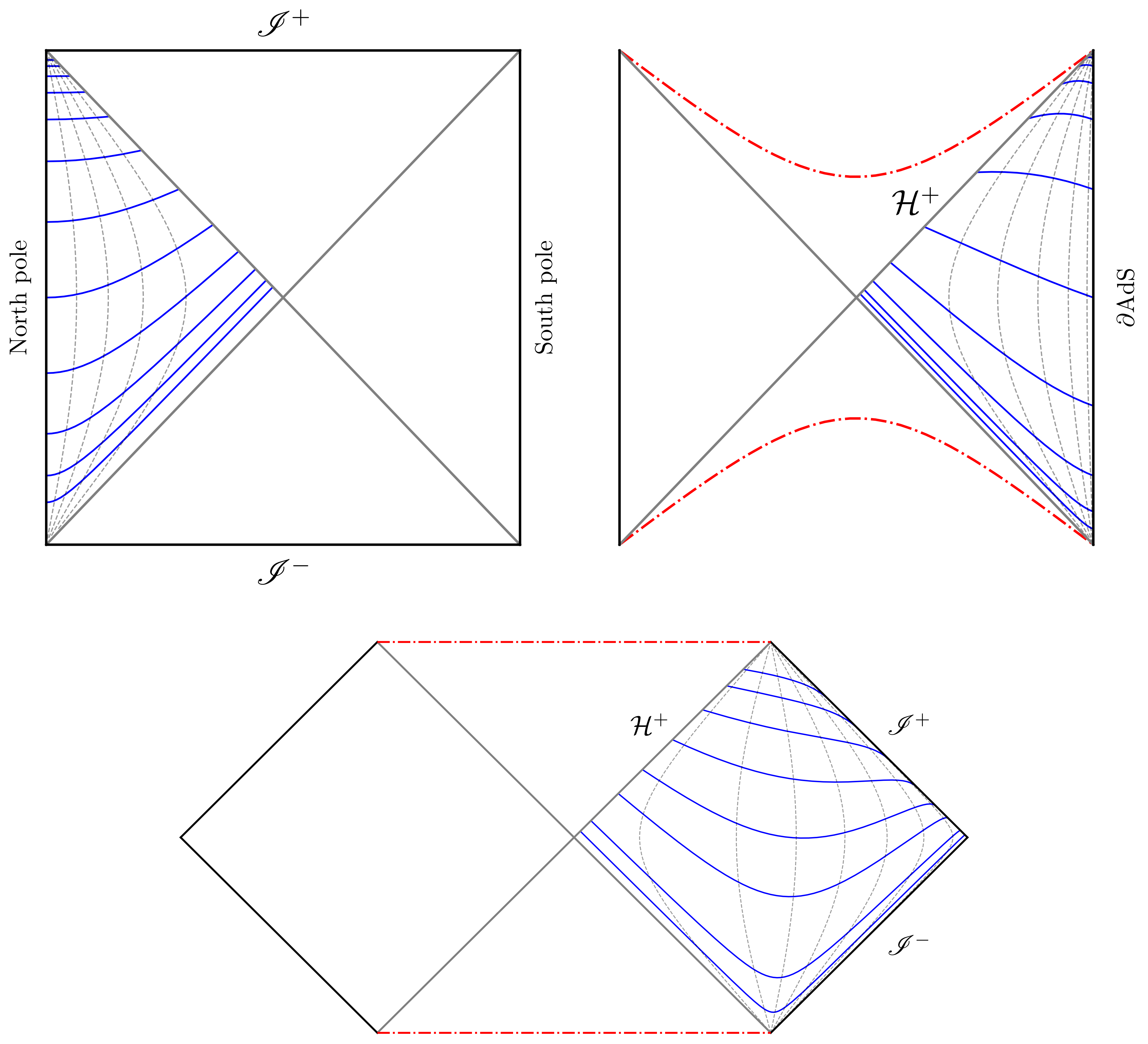
$$h(z) = \frac{1}{2}\log(1-z)$$
$$R(z) = \sqrt{z}$$

- Schwarzschild-AdS_{d+1} black brane:

$$h(z) = \frac{1}{d}\log(1-z)$$
$$R(z) = \frac{1}{z}$$

- Schwarzschild:

$$h(z) = \log(z) + \log(1-z) - 1/z$$
$$R(z) = \frac{1}{z}$$



[JC, Withers '24]

D. Hamiltonian and energy

- Hamiltonian:

$$i\partial_\tau \xi = \mathscr{H}\xi$$

$$\xi = (\phi, \partial_\tau \phi)^T$$

$$\mathscr{H} = \begin{pmatrix} 0 & i \\ \mathscr{L}_1 & \mathscr{L}_2 \end{pmatrix}$$

where

$$\mathscr{L}_1 = \frac{i}{w(z)}\left(p(z)\partial_z^2 + p'(z)\partial_z - q(z)\right)$$

$$\mathscr{L}_2 = \frac{i}{w(z)}\left(2\gamma(z)\partial_z + \gamma'(z)\right)$$

- Energy:

$$\left\langle \xi_1, \xi_2 \right\rangle = \frac{1}{2}\int_0^1 dz \left(w(z)\,\partial_\tau\overline{\phi}_1\partial_\tau\phi_2 + p(z)\,\partial_z\overline{\phi}_1\partial_z\phi_2 + q(z)\,\overline{\phi}_1\phi_2\right)$$

$$E[\xi] = \langle \xi, \xi \rangle$$

E. dS QNMs and sums

- dS_{d+1} QNMs:

$$\phi^{\pm}_{nl}(\tau, z) = e^{-i\omega^{\pm}_{nl}\tau}{}_2F_1\left(-n, \frac{d}{2} - n - \Delta_{\pm}; \frac{d}{2} + l; z\right) \quad \text{where} \quad -\Delta(\Delta - d) = m^2$$

$$\omega^{\pm}_{nl} = -i\left(\Delta_{\pm} + 2n + l\right)$$

- Sum of M modes for $d = 3,\ m^2 = 2,\ l = 0$:

$$\phi(\tau, z) = \sum_{k=1}^{M} (-1)^{1+k} 2^{\frac{3}{2}-k} \sqrt{M(2M-1)} \frac{\Gamma(M)}{\Gamma(k)\Gamma\left(1 - k + M\right)} \phi_k(\tau, z)$$

$$= 2^{\frac{1}{2}-M} \sqrt{\frac{2M-1}{M}} \frac{\left(2 + e^{-\tau}(\sqrt{z} - 1)\right)^M - \left(2 - e^{-\tau}(\sqrt{z} + 1)\right)^M}{\sqrt{z}}$$

$$E(\tau) = 1 - (1 - e^{-\tau})^{2M-1}\left(1 + (2M-1)e^{-\tau}\right) \quad \Rightarrow \quad E(\tau) = 1 - 2M\tau^{2M-1} + O(\tau)^{2M}$$

F. Matrix norm

- Complex matrix A:

$$\|A\| = \max_{x\in\mathscr{V}} \frac{\|Ax\|}{\|x\|} = \max_{x\in\mathscr{V}} \frac{\sqrt{\langle Ax, Ax\rangle}}{\|x\|} = \max_{x\in\mathscr{V}} \frac{\sqrt{\langle x, A^\dagger Ax\rangle}}{\|x\|}$$

- Self-adjoint and $\exists A^{-1}$:

$$A^\dagger A e_i = \lambda_i e_i \quad \to \quad \lambda_i \text{ real positive}$$

$$\langle x, A^\dagger Ax\rangle = \sum_{i,j} \bar{x}^i x^j \langle e_i, A^\dagger A e_j\rangle = \sum_i \bar{x}^i x^i \lambda_i \leq \sum_i \lambda_n \bar{x}^i x^i = \lambda_n \|x\|^2$$

$$\|A\| = \max_{x\in\mathscr{V}} \frac{\|Ax\|}{\|x\|} = \sqrt{\lambda_n} = \sqrt{\max_{\lambda\in\sigma(A^\dagger A)} \{\lambda\}}$$

G. Regge-Wheeler-Zerilli perturbations

- Regge-Wheeler/axial perturbations:

$$q(z) = l(l+1) + (1-s^2)z$$

- Zerilli/polar perturbations:

$$q(z) = \frac{1}{3}(l-1)(l+2)\left(\frac{2(l-1)(l+2)\left(l^2+l+1\right)}{\left(l^2+l+3z-2\right)^2}+1\right)+z$$

H. QR decomposition

- Numerical spectrum and inner product:

$$V_W = \left(\vec{\xi}_1\ \vec{\xi}_2\ \dots\ \vec{\xi}_M\right), \quad G = F^*F$$

- QR decomposition of FV_W gives us orthonormal vectors in energy

$$FV_W = (FQ_W)U_W$$

where $Q_W = \left(\vec{\psi}_1\ \vec{\psi}_2\ \dots\ \vec{\psi}_M\right)$

and U_W is an upper-triangular change of basis matrix