Hyperboloidal coordinates and blowup for wave equations

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Wave maps

- Wave maps $U : (\mathbb{R}^{1,d}, \eta) \to (M, g)$ are critical points of

$$S(U) = \int_{\mathbb{R}^{1,d}} \eta^{\mu\nu} \partial_\mu U^a \partial_\nu U^b g_{ab} \circ U$$

- Corotational map $U : \mathbb{R}^{1,3} \to \mathbb{S}^3$ has the form

$$U(t, r, \theta, \varphi) = (\psi(t, r), \theta, \varphi)$$

- Wave maps equation for corotational maps $U : \mathbb{R}^{1,3} \to \mathbb{S}^3$ reads

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi(t, r) + \frac{\sin(2\psi(t, r))}{r^2} = 0 \qquad (1)$$

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Finite-time blowup

- Eq. (1) is energy-supercritical
- Eq. (1) has explicit self-similar solution [Shatah 1988, Turok-Spergel 1990]

$$\psi_T(t,r) = 2\arctan\left(\frac{r}{T-t}\right)$$

- Eq. (1) has many self-similar solutions $\psi(t,r) = f_n(\frac{r}{T-t})$ [Bizoń 2000]
- Solution ψ_T is conjectured to provide the *generic* blowup profile [Bizoń-Chmaj-Tabor 2000]
- No type II blowup [Dodson-Lawrie 2015]

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Stability of blowup

- Solution ψ_T is nonlinearly asymptotically stable under small perturbations of the initial data [D. 2011, D.-Schörkhuber-Aichelburg 2012, Costin-D.-Xia 2014, Costin-D.-Glogić 2017, Glogic 2022, D.-Wallauch 2023]
- More precisely: Consider Cauchy problem

$$\begin{cases} \left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi(t,r) + \frac{\sin(2\psi(t,r))}{r^2} = 0 \\ \psi(0,\cdot) = \psi_1(0,\cdot) + f, \qquad \partial_0\psi(0,\cdot) = \partial_0\psi_1(0,\cdot) + g. \end{cases}$$

- $\||\cdot|^{-1}f(|\cdot|)\|_{H^3(\mathbb{B}^5_{1+\delta})} + \||\cdot|^{-1}g(|\cdot|)\|_{H^2(\mathbb{B}^5_{1+\delta})} \ll \delta$
$\Rightarrow \exists T \approx 1$ such that

$$(T-t)^{\frac{3}{2}-k}\left\||\cdot|^{-1}\left(\psi(t,|\cdot|) - \psi_T(t,|\cdot|)\right)\right\|_{\dot{H}^k(\mathbb{B}^5_{T-t})} \le \delta(T-t)^{\epsilon}$$

for $k = 0, 1, 2, 3$

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Formulation as a semlinear wave equation

- Weighted Sobolev norms in 5 dimensions because $\widehat{u}(t,r) = \frac{\psi(t,r)}{r}$ satisfies

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\widehat{u}(t,r) = \frac{2r\widehat{u}(t,r) - \sin(2r\widehat{u}(t,r))}{r^3}$$

- In terms of $u(t,x) = \widehat{u}(t,|x|)$,

$$\left(\partial_t^2 - \Delta_x\right)u(t,x) = F(u(t,x),x)$$

with F smooth and $u : I \times \mathbb{R}^5 \to \mathbb{R}$ radial

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Major open question

Is it possible to continue the solution beyond blowup?

Continuation beyond blowup

- Blowup solution ψ_T has geometric interpretation as an angle:

$$\psi_T(t,r) = 2\arctan\left(\frac{r}{T-t}\right) = 2\arg(T-t+ir)$$

- Compact expression for $\arg$:

$$\arg z = 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z + \sqrt{\operatorname{Re}^2 z + \operatorname{Im}^2 z}}\right)$$

- More general blowup solution

$$\psi_T^*(t,r) := 4\arctan\left(\frac{r}{T-t+\sqrt{(T-t)^2+r^2}}\right)$$

solves Eq. (1) for all $t \in \mathbb{R}$ and $r > 0$ and $\psi_T^*(t,r) = \psi_T(t,r)$ for $t < T$

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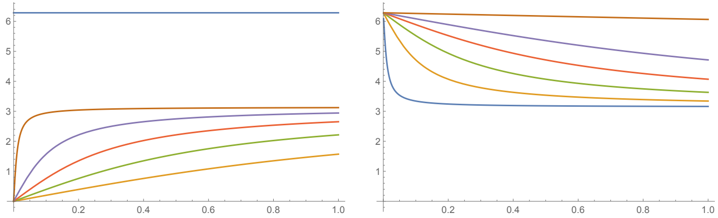
Properties of ψ_T^*

- We have $\psi_T^*(t,0) = 0$ for $t < T$ but

$$\lim_{r \to 0+} \psi_T^*(t,r) = 2\pi$$

for $t > T$ $\rightsquigarrow$ boundary condition changes at blowup

- $\lim_{t\to\infty} \psi_T^*(t,r) = 2\pi$
- Snapshots of evolution



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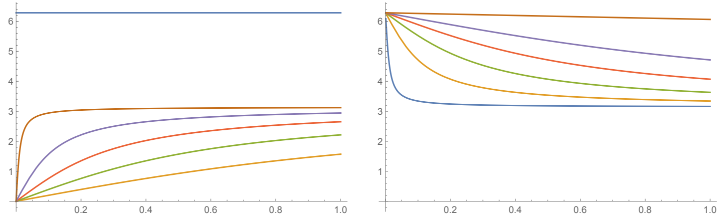

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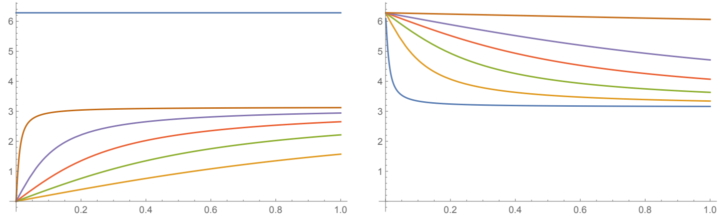

Similarity coordinates

- Previous results rely heavily on *similarity coordinates* (τ, ξ)

$$\tau = -\log(T-t), \qquad \xi = \frac{x}{T-t}$$

or

$$t = T - e^{-\tau}, \qquad x = e^{-\tau}\xi$$

- Similarity coordinates are natural for studying evolution in backward lightcone but are unable to cover the whole space
- Geometric problem: Slices $\tau = \text{const}$ are flat

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Hyperboloidal similarity coordinates

- HSC (s, y) are generalizations of similarity coordinates defined by
$$t = T + e^{-s}h(y), \qquad x = e^{-s}y$$
with $h(y) = \sqrt{2 + |y|^2} - 2$
- Slices $s = \mathrm{const}$ are hyperboloids
- HSC are compatible with self-similarity: $\frac{x}{T-t} = -\frac{y}{h(y)}$
- HSC cover a large portion of spacetime almost up to the Cauchy horizon of the singularity

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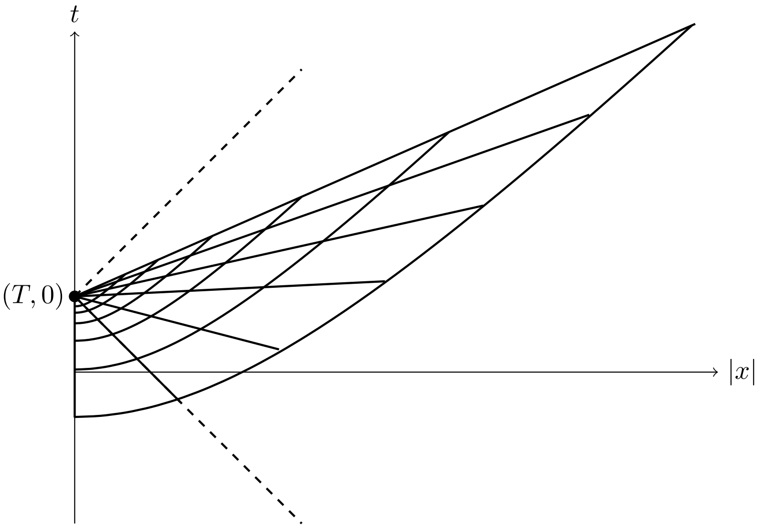
Geometry of HSC



- Start with half-wave equation $\partial_t u_-(t,x) - \partial_x u_-(t,x) = 0$
- In HSC:

$$[1 - h'(y)]\partial_s v_-(s,y) = -[y - h(y)]\partial_y v_-(s,y),$$

  where $v_-(s,y) = u_-(T + e^{-s}h(y), e^{-s}y)$
- Test with v_- and integrate:

$$\|v_-(s,\cdot)\|_{L^2(\mathbb{B}_R)} \lesssim e^{s/2}\|v_-(s_0,\cdot)\|_{L^2(\mathbb{B}_R)}$$

  provided $R > \frac{1}{2}$
- More general:

$$\|v_-(s,\cdot)\|_{H^k(\mathbb{B}_R)} \lesssim e^{s/2}\|v_-(s_0,\cdot)\|_{H^k(\mathbb{B}_R)}$$

- Analogously:

$$\|v_+(s,\cdot)\|_{H^k(\mathbb{B}_R)} \lesssim e^{s/2}\|v_+(s_0,\cdot)\|_{H^k(\mathbb{B}_R)}$$

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Control of 1d wave evolution in HSC (continued)

- Suppose

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0 &= (\partial_t^2 - \partial_x^2)u(t,x) = (\partial_t - \partial_x)(\partial_t + \partial_x)u(t,x) \\
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\end{aligned}
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$\Rightarrow u_\pm(t,x) := (\partial_t \mp \partial_x)u(t,x)$ satisfy half-wave equations

- In HSC, for $v(s,y) = u(T + e^{-s}h(y), e^{-s}y)$,

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\partial_s v(s,y) &= \tfrac{1}{2}e^{-s}[y - h(y)]v_-(s,y) - \tfrac{1}{2}e^{-s}[y + h(y)]v_+(s,y) \\
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- If $v(s,\cdot)$ is odd,

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\begin{aligned}
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$$\partial_s v(s,y) = \tfrac{1}{2}e^{-s}[y - h(y)]v_-(s,y) - \tfrac{1}{2}e^{-s}[y + h(y)]v_+(s,y)$$
$$\partial_y v(s,y) = -\tfrac{1}{2}e^{-s}[1 - h'(y)]v_-(s,y) + \tfrac{1}{2}e^{-s}[1 + h'(y)]v_+(s,y)$$

- If $v(s,\cdot)$ is odd,

$$\|v(s,\cdot)\|_{H^k(\mathbb{B}_R)} + \|\partial_s v(s,\cdot)\|_{H^{k-1}(\mathbb{B}_R)}$$
$$\lesssim e^{-s/2}\left(\|v(s_0,\cdot)\|_{H^k(\mathbb{B}_R)} + \|\partial_0 v(s_0,\cdot)\|_{H^{k-1}(\mathbb{B}_R)}\right)$$

Semigroup formulation

- Write 1d wave equation in HSC as

$$\partial_s \begin{pmatrix} v(s,\cdot) \\ \partial_s v(s,\cdot) \end{pmatrix} = \mathbf{L}_1 \begin{pmatrix} v(s,\cdot) \\ \partial_s v(s,\cdot) \end{pmatrix}$$

- Lumer-Phillips: $\mathbf{L}_1$ generates semigroup $\mathbf{S}_1(s)$ on $H^k(\mathbb{B}_R) \times H^{k-1}(\mathbb{B}_R)$ with

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Control of 5d radial wave evolution

- 5d radial wave equation

$$\left(\partial_t^2 - \partial_r^2 - \tfrac{4}{r}\partial_r\right)\widehat{u}(t,r) = 0$$

can be reduced to 1d via

$$\partial_r^2(r^2\partial_r + 3r) = (r^2\partial_r + 3r)(\partial_r^2 + \tfrac{4}{r}\partial_r)$$

- Translate this to HSC: Construct operator $\mathbf{D}_5$ such that $\mathbf{D}_5\mathbf{L}_5 = \mathbf{L}_1\mathbf{D}_5 + \mathbf{D}_5$ and

$$\|\mathbf{D}_5\mathbf{f}\|_{H^k(\mathbb{B}_R)\times H^{k-1}(\mathbb{B}_R)} \simeq \|\mathbf{f}\|_{H^{k+1}_{\text{rad}}(\mathbb{B}^5_R)\times H^k_{\text{rad}}(\mathbb{B}^5_R)}$$

- Semigroup $\mathbf{S}_5(s) := e^s\mathbf{D}_5^{-1}\mathbf{S}_1(s)\mathbf{D}_5$ is solution operator to $5d$ equation

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Wave maps in HSC

- Recall wave maps equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\widehat{u}(t,r) = \frac{2r\widehat{u}(t,r) - \sin(2r\widehat{u}(t,r))}{r^3}$$

- In HSC, for $v(s,y) = e^{-s}\widehat{u}(T + e^{-s}h(y), e^{-s}|y|)$,

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- Blowup solution

$$\widehat{u}_T^*(t,r) = \frac{4}{r}\arctan\left(\frac{r}{T - t + \sqrt{(T-t)^2 + r^2}}\right)$$

in HSC:

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Perturbations of the blowup solution

- Ansatz $v(s,y) = v_T^*(s,y) + \phi(s,y)$ yields equation

$$\partial_s \Phi(s) = (\mathbf{L}_5 - \mathbf{I} + \mathbf{L}')\Phi(s) + \mathbf{N}(\Phi(s))$$

for $\Phi(s) = (\phi(s,\cdot), \partial_s\phi(s,\cdot))$

- From $\|\mathbf{S}_5(s)\|_{H^{k+1}(\mathbb{B}^5_R)\times H^k(\mathbb{B}^5_R)} \lesssim e^{s/2}$ we have

$$\sigma(\mathbf{L}_5 - \mathbf{I}) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \leq -\tfrac{1}{2}\}$$

- Operator $\mathbf{L}'$ is compact

$$\Rightarrow \sigma(\mathbf{L}_5 - \mathbf{I} + \mathbf{L}') \setminus \sigma(\mathbf{L}_5 - \mathbf{I}) \subset \sigma_p(\mathbf{L}_5 - \mathbf{I} + \mathbf{L}')$$

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Solution of the spectral equation

- Recall standard similarity coordinates

$$t = T - e^{-\tau}, \qquad x = e^{-\tau}\xi$$

- Relation to HSC:

$$(\tau, \xi) = (s - \log(-h(y)), -\tfrac{y}{h(y)})$$

- Mode solutions are mapped to mode solutions:

$$e^{\lambda\tau}\mathbf{f}_\lambda(\xi) = e^{\lambda s}[-h(y)]^{-\lambda}\mathbf{f}_\lambda(-\tfrac{y}{h(y)})$$

$\Rightarrow$ {eigenvalues in (s, y)} $\subset$ {eigenvalues in (τ, ξ)}

- Eigenvalue problem in (τ, ξ) is solved [Costin-D.-Xia 2014, Costin-D.-Glogić 2017] $\Rightarrow 1 \in \sigma_p(\mathbf{L}_5 - \mathbf{I} + \mathbf{L}')$ and

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Spectral analysis

- Eigenfunction $\mathbf{f}_1^*$ to eigenvalue 1 is explicit, from the fact that $\widehat{u}_T^*$ is one-parameter family of solutions
- Riesz projection

$$\mathbf{P} = \frac{1}{2\pi i} \int_\gamma (\lambda - \mathbf{L}_5 + \mathbf{I} - \mathbf{L}')^{-1} d\lambda$$

  has rank one (ODE argument), i.e., $\operatorname{rg} \mathbf{P} = \langle \mathbf{f}_1^* \rangle$
- Simple resolvent estimate + Gearhart-Prüss: $\exists\, \omega_0 > 0$ such that

$$\|\mathbf{S}(s)(\mathbf{I} - \mathbf{P})\|_{H^{k+1}(\mathbb{B}^5_R) \times H^k(\mathbb{B}^5_R)} \lesssim e^{-\omega_0 s}$$

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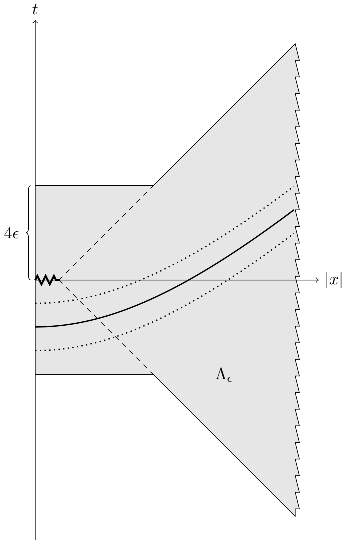
Construction of initial data



Hyperboloidal evolution

- Solution of wave maps equation via fixed point of map

$$\mathbf{K}_{f,g}(\Phi)(s) = \mathbf{S}(s - s_0)\mathbf{U}((f,g),T) + \int_{s_0}^{s} \mathbf{S}(s - s')\mathbf{N}(\Phi(s'))ds'$$

- Nonlinearity $\mathbf{N}$ is locally Lipschitz
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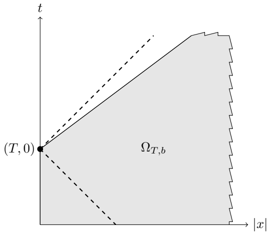
The main theorem

- (f, g) smooth, radial, $\|(f,g)\|_{H^m(\mathbb{R}^5)\times H^{m-1}(\mathbb{R}^5)} \ll \delta$, $\operatorname{supp}(f,g) \subset \mathbb{B}^5_\epsilon$
- $\Rightarrow \exists\, T \in [1-\delta, 1+\delta]$ and a unique smooth solution u with data $u(0,x) = u_1^*(0,x) + f(x)$, $\partial_0 u(0,x) = \partial_0 u_1^*(0,x) + g(x)$ in the domain $\Omega_{T,b}$



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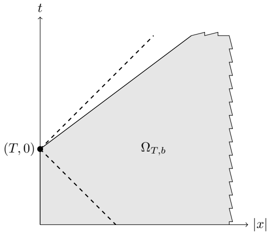

The main theorem (continued)

- Solution u converges to u_T^* in the sense that

$$e^{-s}\|(u\circ\eta_T)(s,\cdot)-(u_T^*\circ\eta_T)(s,\cdot)\|_{H^{m-3}(\mathbb{B}^5_R)}\leq \delta e^{-\omega_0 s}$$
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Thank you

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