Virtual Infinity Seminar

Pseudospectrum of black hole quasi-normal modes: results and obstacles

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- Phys.Rev.D 107 (2023) 6, 064012;
- 2312.11998 (also to appear in PRD)

Talk outline:

- Hyperboloidal (and null) formulation of the QNM problem.
- Norm, QNM instability, and pseudospectrum.
- (Non-)convergence of the pseudospectrum.
- Applications to dynamical stability analysis (of exotic compact objects).

Geometries and perturbations

We will be dealing with (electro-)vacuum geometries in spherical symmetry, with or without Λ,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{1}$$

with $f(r)$ the redshift function. E.g. for Schwarzschild-Anti-de Sitter (SAdS),

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{R^2}.$$

Linear perturbations on this spacetime follow

$$-\partial_{tt}\psi + \partial_{r_* r_*}\psi - V(r)\psi = 0, \tag{2}$$

where the form of $V(r)$ depends on the nature of the perturbation and the angular multipole, and $dr_* = dr/f(r)$.

QNM boundary conditions

QNMs are the (analytic) solutions which satisfy appropriate boundary conditions, such as:

- Ingoing condition at black hole (BH) horizons,

$$(\psi_{,t} - \psi_{,r_*})|_{r_* \to -\infty} = 0. \tag{3}$$

- Outgoing condition at infinity or at cosmological horizon,

$$(\psi_{,t} + \psi_{,r_*})|_{r_* \to \infty} = 0. \tag{4}$$

- On timelike boundaries, such as the AdS asymptotic region or the surface of some exotic compact object (ECO) models, e.g. a Dirichlet condition,

$$\psi|_{\text{boundary}} = 0. \tag{5}$$

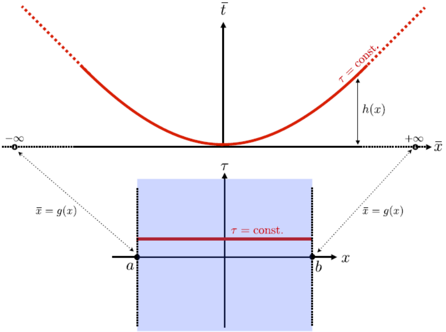
Hyperboloidal slicing

- The solutions at $r_* \to \pm\infty$ are the modes $e^{i\omega(t \mp r_*)}$.
- On a $t =$ const. slice, if they are decaying in time ($\mathrm{Im}(\omega) > 0$), then the modes diverge in their respective limits of r_* (at the horizon bifurcation surface and at spacelike infinity).
- The solutions are only regular in r_* at the boundaries if $t \to \infty$ in a compensatory manner.

Switch to hyperboloidal time τ, and compactified radius χ:

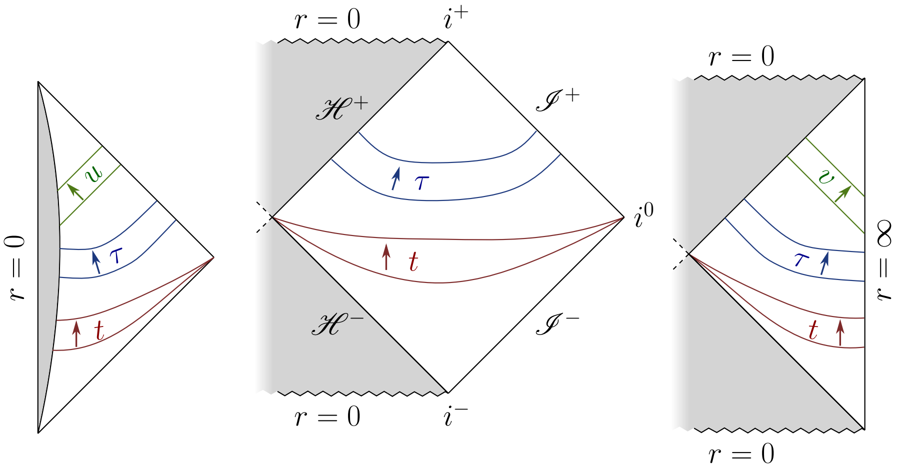
$$t = \tau - h(\chi),$$
$$r_* = g(\chi),$$

such that $h \underset{r_* \to -\infty}{\sim} g$, and $h \underset{r_* \to \infty}{\sim} -g$. When possible, τ can also be an ingoing ($h = g$) or outgoing ($h = -g$) **null coordinate**.



From Jaramillo et al., *Phys. Rev. X 11, 031003* (2021).

Slicings



$r = 0$
u
τ
t
$r = 0$
i^+
$\mathscr{H}^+$
$\mathscr{I}^+$
τ
i^0
t
$\mathscr{H}^-$
$\mathscr{I}^-$
$r = 0$
i^-
$r = 0$
v
τ
t
$r = \infty$
$r = 0$

QNMs as eigenvalue problems

- In the hyperboloidal coordinate system $\{\tau, \chi\}$, after an order-reduction in time and a Fourier transform, the wave eq. becomes an **eigenvalue problem**,

$$L \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \omega \begin{pmatrix} \psi \\ \phi \end{pmatrix},$$

with $\phi = \partial_\tau \psi$, and

$$L = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ L_1(\chi, \partial_\chi) & L_2(\chi, \partial_\chi) \end{pmatrix}.$$

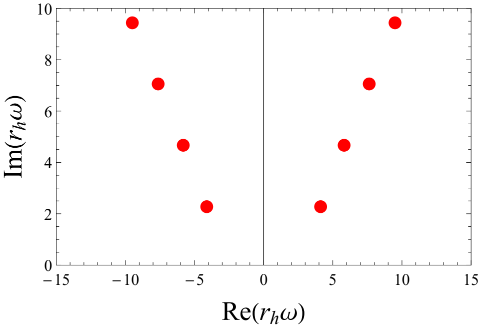
- In a null coordinate system, after a Fourier transform, the equation becomes a **generalised eigenvalue problem**,

$$M\psi = \omega B \psi$$

with $M = M(\chi, \partial_\chi)$ and $B = B(\chi, \partial_\chi)$.

QNMs: numerical computation

After a discretisation in χ on a Chebyshev grid and a pseudospectral approximation to ∂_χ, we can solve the problem numerically.



QNM frequency spectrum for a scalar field, $\ell = 2$, Schwarzchild-Anti-de Sitter (SAdS) with $r_h = R_{AdS}$.

Perturbations to L

If we perturb L, the eigenvalues change.[1]

$$L \to \tilde{L} = L + \delta L,$$

Size of δL: energy norm,[2] coming from the product

$$\langle u_1, u_2 \rangle_E = \frac{1}{2}\int_a^b \left[w(\chi)\bar{\phi}_1\phi_2 + p(\chi)\partial_\chi\bar{\psi}_1\partial_\chi\psi_2 + q(\chi)\bar{\psi}_1\psi_2 \right] d\chi.$$

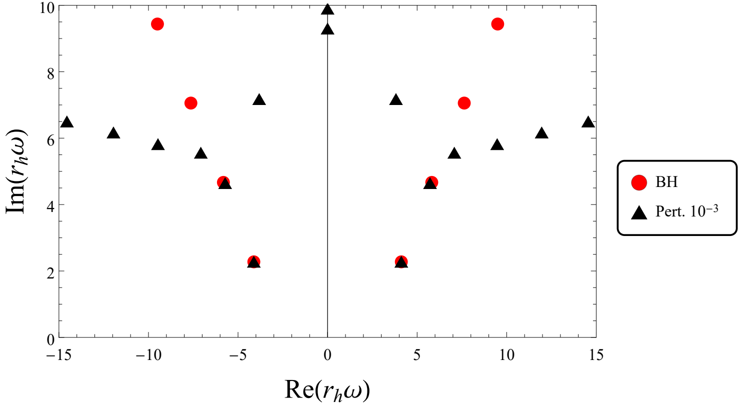
For $\|\delta L\| = \epsilon$, if the migration of any of the QNMs $|\tilde{\omega}_n - \omega_n|$ is greater than ϵ, then the operator L is **spectrally unstable** in this norm. This occurs with BH QNMs.

We consider a "physical" perturbation one which affects only the potential V (without changing its boundary behaviour).

[1] Jaramillo, Panosso Macedo & Al Sheikh, *Phys. Rev. X 11, 031003* (2021).

[2] Gasperin & Jaramillo, *Class. Quantum Grav. 39 115010* (2022).

QNM instability



Scalar field, $\ell = 2$, Schwarzchild-Anti-de Sitter (SAdS) with $r_h = R_{AdS}$.
Spectrum before and after a perturbation $\delta\tilde{V} = \varepsilon\,\sin(2\pi k\chi)$.

Nollert-Price branches [Nollert & Price, *J. Math. Phys.* 40, 980–1010 (1999)]

Pseudospectrum

The QNM spectrum is given by

$$\sigma(L) = \{\omega \in \mathbb{C} : |L - \omega \mathbb{I}| = 0\}. \tag{6}$$

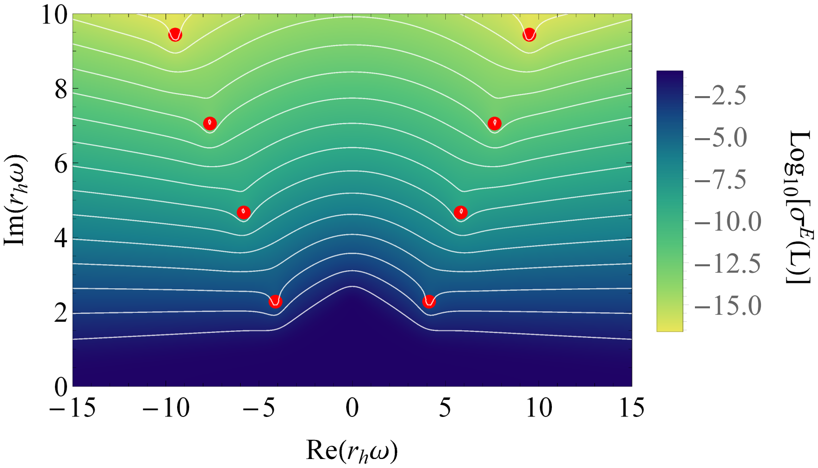
The ϵ-pseudospectrum is a region in $\mathbb{C}$ in which all points are no further than ϵ from being eigenvalues in the following sense:

$$\sigma^{\epsilon}(L) = \{\lambda \in \mathbb{C} : \|R_L(\lambda)\| = \|(L - \lambda \mathbb{I})^{-1}\| > 1/\epsilon\}, \tag{7}$$

where $R_L(\lambda)$ is called the *resolvent*.

- When this measure of closeness coincides with actual distance in $\mathbb{C}$, the operator is **spectrally stable** (and the ϵ-pseudospectral regions form concentric discs around each point of the spectrum).
- Conversely, when the ϵ-pseudospectrum contains points further than a distance ϵ away from the closest point of the spectrum, the operator is **spectrally unstable** in the chosen norm.

Pseudospectrum



Pseudospectrum of the same case as above, calculated with $\|R_L(\lambda)\|$.

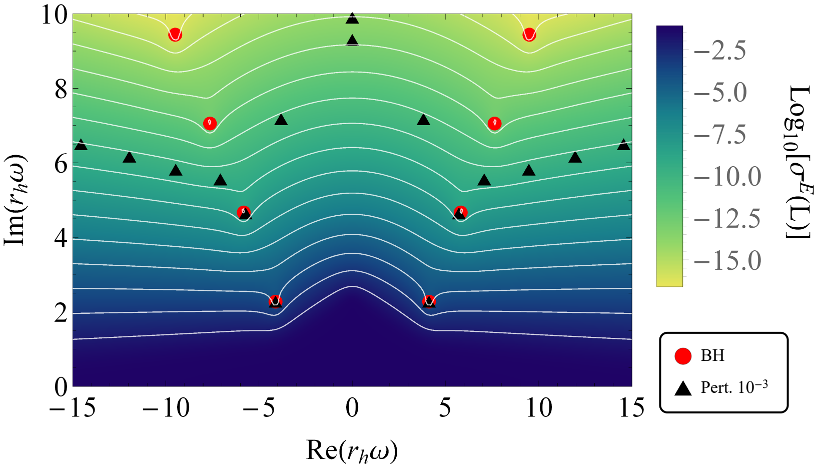
Pseudospectrum and operator perturbations

An equivalent definition of the ϵ-pseudospectrum is

$$\sigma^{\epsilon}(L) = \{\lambda \in \mathbb{C}, \exists \delta L, \|\delta L\| < \epsilon : \lambda \in \sigma(L + \delta L)\}, \tag{8}$$

where δL is any perturbation to L. After a perturbation of magnitude ϵ, the modes can end up anywhere inside the ϵ-pseudospectral region.

Pseudospectrum and operator perturbations



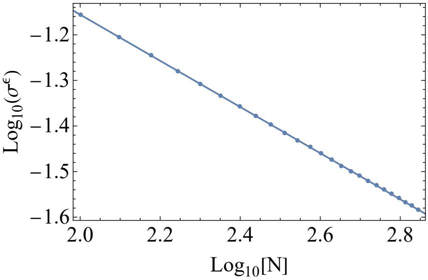
Pseudospectrum calculated with $\|R_L(\lambda)\|$, and perturbed modes after δL.

Pseudospectrum and norm

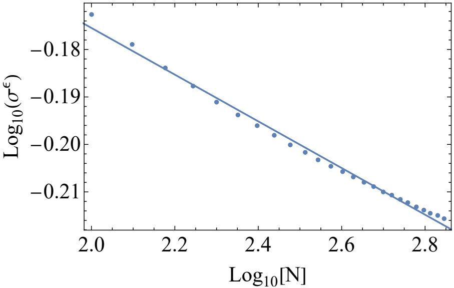
- Both definitions **depend on the choice of norm**.
- In mathematical problems, when faced with spectral instability in one norm, one usually looks for another in which stability is recovered.
- In physics, however, the choice of “large” and “small” must be grounded in a measurable quantity, such as energy.

(Non-)convergence of the pseudospectrum

This would be the full picture, but there is an issue with numerical convergence...



Energy norm of (inverse of) resolvent operator at $\lambda = 2 + 0.5i$ (for same SAdS problem as above) as function of numerical resolution N. Similar whenever $\mathrm{Im}(\lambda) > 0$.



Energy norm of (inverse of) resolvent operator at $\lambda = 2 - 0.5i$ (for same SAdS problem as above) as function of numerical resolution N. Similar whenever $\mathrm{Im}(\lambda) < 0$.

Qualitative and quantitative aspects

Observations:

- The spectral instability under "small" perturbations is a robust result
- The pseudospectrum at a finite numerical resolution N captures the qualitative behaviour of the spectral instability
- However, the ϵ values assigned to each contour line change with N

Possible reasons for non-convergence:

- The operator cannot be numerically approximated by finite-rank matrices
- The points with non-convergence are actually **part of the spectrum**...?

Way forward

- Analyse convergence with the numerical gridpoint number N.
- Understand the reason behind non-convergence.
- Determine when and how quantitative results can be obtained.

[e.g. Boyanov, Cardoso, Destounis, Jaramillo, Panosso-Macedo: arXiv:2312.11998]

- Analyse qualitative aspects and implications of the non-convergent result.
- Obtain quantitative results from the $\mathrm{Im}(\lambda) < 0$ region (early-time non-modal dynamical effects, nonlinear instabilities).

[e.g. Boyanov, Cardoso, Destounis, Jaramillo, Panosso-Macedo: Phys.Rev.D 107 (2023) 6, 064012]

Pseudospectrum convergence in Schwarzschild-AdS

AdS QNMs: a precise definition

- In a Sobolev functional space H^k, the spectrum of L (for the SAdS BH) is only discrete in the region[3]

$$\mathrm{Im}(\lambda) < a + \kappa(k - \frac{1}{2}), \tag{9}$$

where κ is the surface gravity of the BH horizon, and a is a constant.

- Above this band, all points are part of the spectrum.
- Because part of the spectrum is always continuous, L in H^k is never a compact operator.
- L might not be well approximated by finite-rank matrices.
- Still, improved convergence behaviour might be found in some region of $\mathbb{C}$ if we impose higher regularity.

[3] C. Warnick, *Commun. Math. Phys. 333, 959–1035* (2015).

H^k norms and convergence

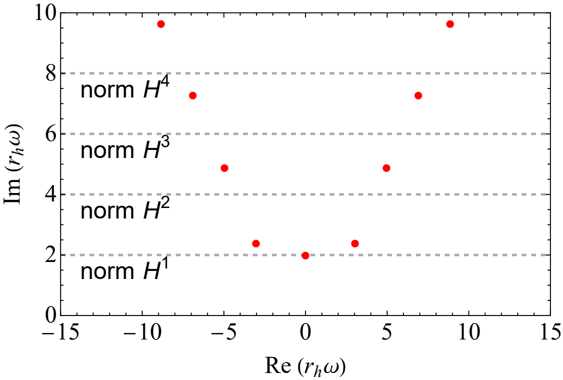
- In practice, the way to ensure the function space is H^k is through the norm, increasing the number of derivatives,

$$\langle u_1, u_2 \rangle_k = \frac{1}{2}\int_a^b \left[w(\chi)\bar{\phi}_1\phi_2 + p(\chi)\partial_\chi\bar{\psi}_1\partial_\chi\psi_2 + q(\chi)\bar{\psi}_1\psi_2 + \partial_\chi^k\bar{\psi}_1\partial_\chi^k\psi_2 \right] d\chi.$$

- Using **the null slicing**, we did obtain such a result (the hyperboloidal slicing lead to some complications).

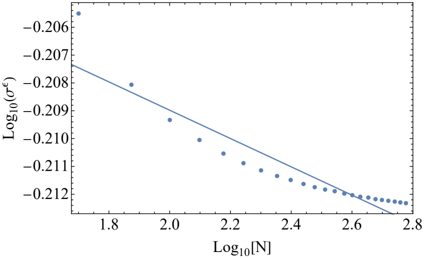
H^k norms and convergence

The result is as shown below:

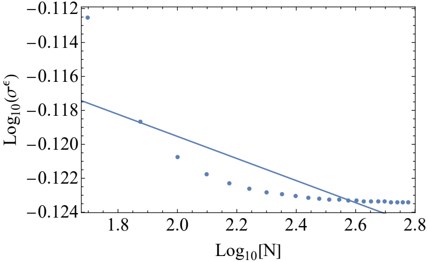


Convergence example

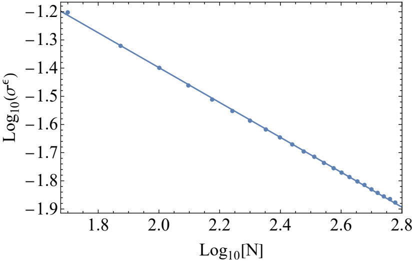
$\lambda = 2 + 1.5i$, norm H^1



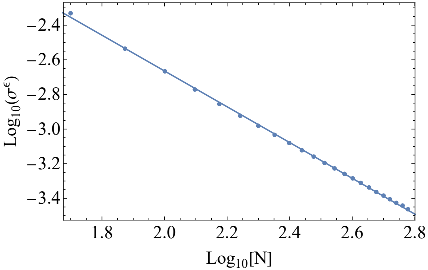
$\lambda = 2 + 3i$, norm H^2



$\lambda = 2 + 2.6i$, norm H^1



$\lambda = 2 + 5i$, norm H^2



Summary

- The pseudospectrum, as defined by the level sets of the norm of the resolvent operator, does not always converge numerically.
- There appears to be a strong relation between its numerical convergence and the degree of regularity encoded in the norm.
- This issue is under active exploration, both for AdS BHs, as well as for other QNM problems.

Exotic compact object dynamical stability analysis

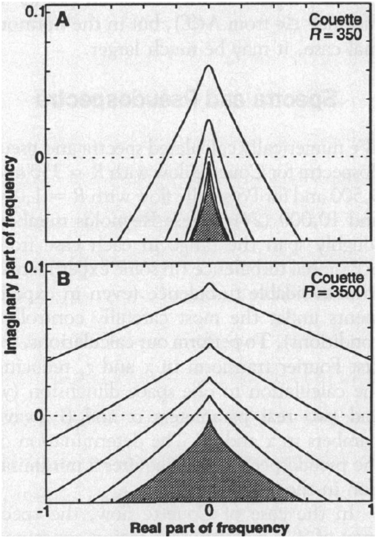
Pseudospectrum: other uses

- Quantifying spectral instability is not the only use of the pseudospectrum
- It also contains information on **transient non-modal behaviour** in time domain evolution, and **pseudo-resonant amplification** of sources
- In this way, it can be used as a tool to gauge the **dynamical instability** of systems when it is not directly evident in the spectrum
- As we will see, this information is obtained from the region at and below the real axis, where the energy norm suffices for convergence.

Spectral instability in hydrodynamics

Linear perturbations around stationary flow:[4]

- No growing modes
- Spectral instability
- Long-lived modes – very close to real axis (close the higher R is)
- Pseudospectral contour lines expand to the unstable (growing mode) half of the complex plane
- Full non-linear evolution shows instability

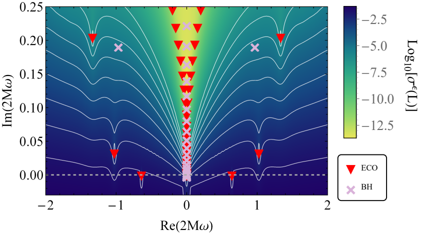


[4] Trefethen et al., *Hydrodynamic stability without eigenvalues*, Science 261, 578 (1993).

Pseudospectrum of an ECO with surface at $\mathcal{E} = 10^{-3}$

Linear perturbations on an ECO with a reflective surface:

- No growing modes
- Spectral instability
- Long-lived modes – very close to real axis (closer the more compact the object is)
- Pseudospectral contours expand to the "unstable" (growing mode) half of the complex plane
- Full non-linear evolution shows instability[5]



Pseudospectrum of an $\ell = 2$ scalar perturbation on an ECO with a reflective surface at $r = (1 + \mathcal{E})2M$, with $\mathcal{E} = 10^{-3}$.

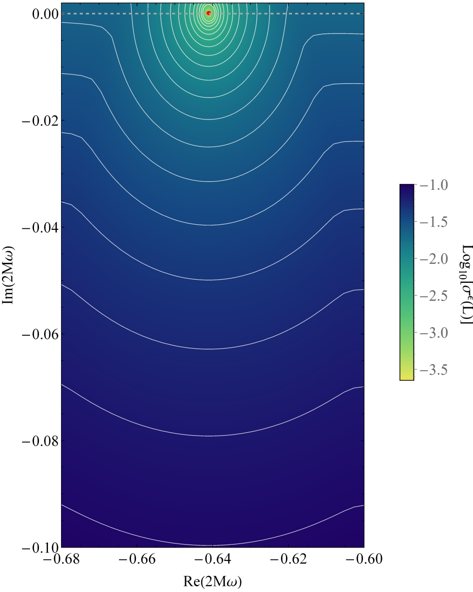
[5] Dailey, Afshordi, Schnetter, *Reflecting boundary conditions in numerical relativity as a model for black hole echoes*, arXiv:2301.05778

Fundamental mode zoom

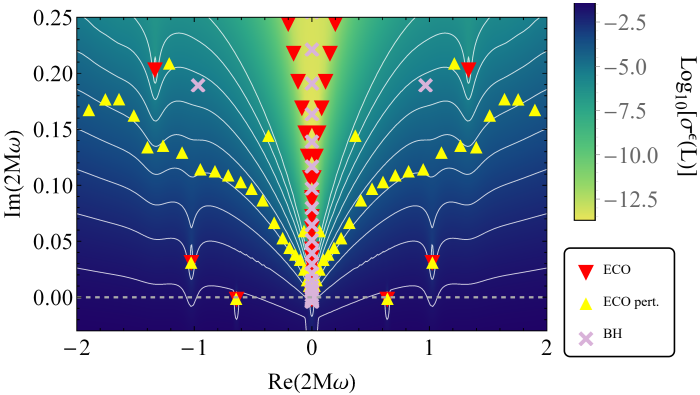
The protrusion of the contour lines to the “unstable” side could imply that:

- physically perturbing system leads to **modes which grow**
- initially small solutions have a period of **transient growth**
- small sources could be **pseudo-resonantly amplified**

All three of these are present in the hydrodynamic system mentioned. But what about ECOs?



Perturbed ECO spectrum



No unstable modes after physical perturbation (adding noise to the discretised effective potential, magnitude $\|\delta L\| \sim 10^{-1}$).

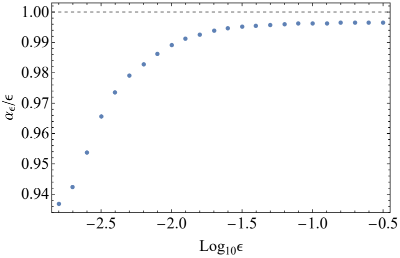
Transient evolution bound

Transient growth: norm of the evolution operator $\|e^{iL\tau}\|$ bigger than 1?

This norm is bounded by the *Kreiss constant*,

$$\mathcal{K}(L) = \lim_{\epsilon\to\infty} \frac{\alpha_\epsilon(L)}{\epsilon}, \quad \text{with} \quad \alpha_\epsilon(L) = -\inf_{\lambda\in\sigma^\epsilon(L)} \mathrm{Im}(\lambda). \tag{10}$$

Numerically we find that $\alpha_\epsilon(L)/\epsilon \to 1$,



which indicates that the evolution operator $e^{iL\tau}$ is contractive (**no transient growth**).

Pseudoresonances

The final possibility: pseudoresonances

$$(\partial_\tau - iL)\, u^{(1)} = S^{(1)} \tag{11}$$

The maximal amplification of a source with frequency Ω is given by

$$\|(L - \Omega\mathbb{I})^{-1}\|,$$

coinciding with the prescription for the pseudospectrum at $\lambda = \Omega$.
However, the instability is expected even when $S^{(1)} = 0$.

As it turns out, in an appropriate gauge the equation for the second order perturbation is[6]

$$(\partial_\tau - iL)\, u^{(2)} = S^{(2)}[u^{(1)}]. \tag{12}$$

A Fourier transform decomposes $S^{(2)}$ into modes $e^{i\Omega t}$, with Ω proportional to (the real part of) the frequencies of $u^{(1)}$.

$\rightarrow$ **Pseudoresonance between different orders** (breakdown of perturbation theory)

[6] J. L. Jaramillo, Class.Quant.Grav. 39 (2022) 21, 217002

Summary (part 2)

- The pseudospectrum of the linear evolution operator can be used to gauge the presence of non-linear instabilities.
- The dynamical instability of ECOs is characterised by neither a displacement of modes to the unstable half of the complex plane after a perturbation, nor by non-modal transient growth (as occurs e.g. in hydrodynamics).
- Rather, **pseudoresonances** can be produced between different orders in perturbation theory. The closer the QNMs are to the real axis (i.e. the more compact the ECO), the more likely it is for this phenomenon to lead to a breakdown in the perturbation theory.

Outlook

- The pseudospectrum of QNMs can be used to gain insight into a variety of problems in gravitational physics.
- Due to non-convergence in the energy norm, its use is not as straightforward as might have been expected, though this leads to other open questions (do we change our definition of QNM spectrum, or our notion of a physical norm?).
- All projects mentioned are still works in progress, and there are many other open problems still (e.g. the pseudospectrum beyond spherical symmetry).

Thank you for your attention!