Asymptotics in General Relativity

The role of spatial infinity

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Overview

Key ideas in this talk

- The relation between the asymptotic behaviour of the gravitational at null infinity and spatial infinity —the **problem of spatial infinity**.
- **Penrose's conditions** to study isolated systems in General Relativity are **too restrictive** to describe generic spacetimes.
- A conformal approach to the structure of spatial infinity by H. Friedrich paves the way to **a full understanding of the relation between Cauchy data and the asymptotic behaviour of the gravitational field** —thus settling the problem of spatial infinity.
- Applications of these ideas to the **computation of asymptotic charges**.

Introduction

Asymptopia

Asymptopia

A far away land of which we know little… (JM Stewart)

- There is a vast literature on the ***asymptotics of the gravitational field***.
- Builds on Penrose’s characterisation of isolated systems in GR using the notion of ***asymptotic simplicity***.
- Most of it formal: it makes a number of assumptions which ***may or may not be generic.***

Understanding the assumptions

A vast body of work aimed at setting the asymptotics of GR on a solid footing

- H Friedrich, JAVK,…
- P Chrusciel, R Beig & BG Schmidt,…
- D Christodoulou & S Klainermann, Klainermann & Nicolo, Lindblad & Rodnianski,…
- P Hintz & A Vasy
- L Kehrberger,…

Although great progress has occurred in recent years, some work is still required!

Not comprehensive!

Some (historical) context

Asymptotic simplicity (AS) R Penrose 1963-65

Definition 7.1 (***asymptotically simple spacetimes***) *A spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to be* ***asymptotically simple*** *if there exists a smooth, oriented, time-oriented, causal*[1] *spacetime* $(\mathcal{M}, \boldsymbol{g})$ *and a smooth function* Ξ *on* $\mathcal{M}$ *such that:*

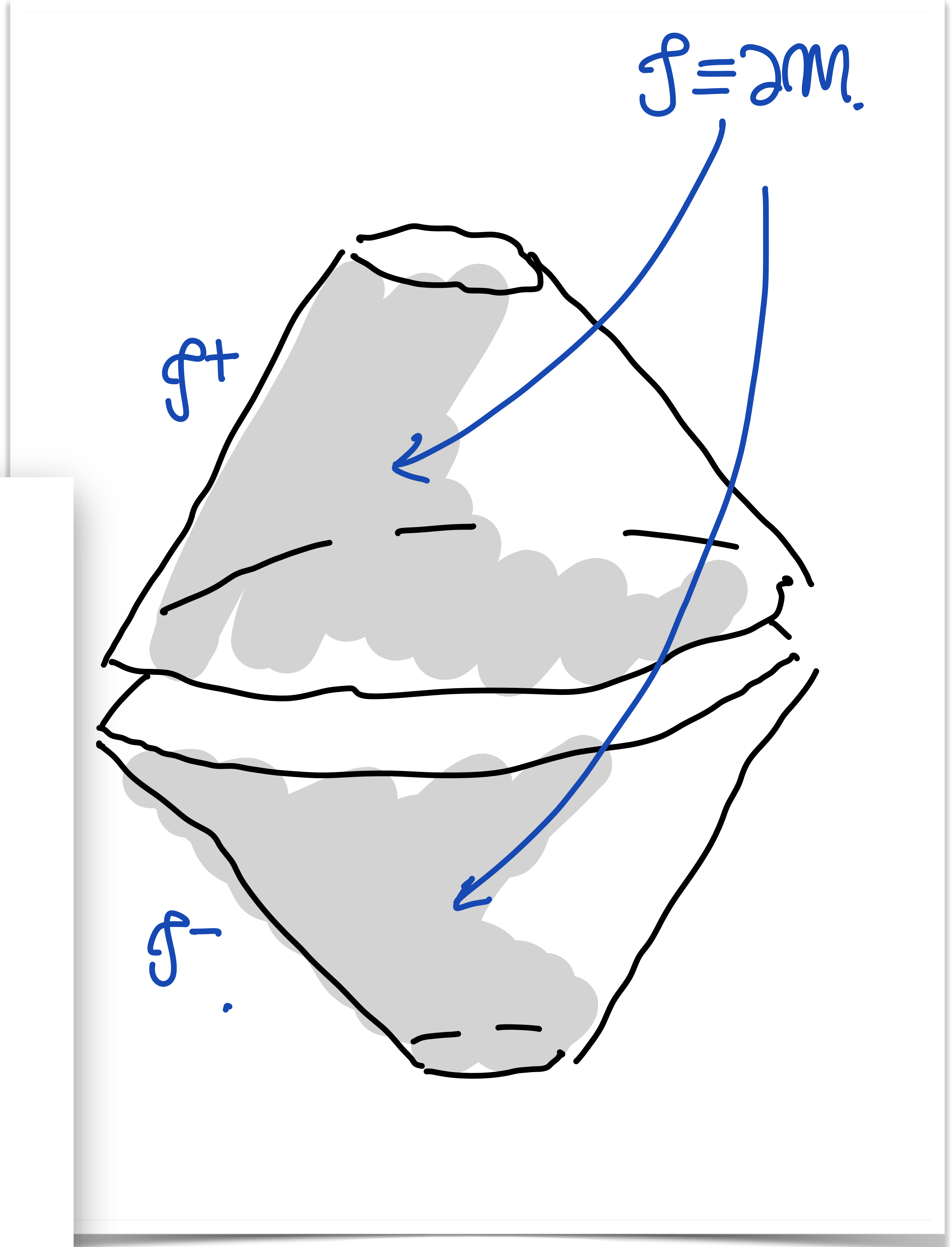
(i) $\mathcal{M}$ *is a manifold with boundary* $\mathscr{I} \equiv \partial\mathcal{M}$.

(ii) $\Xi > 0$ *on* $\mathcal{M} \setminus \mathscr{I}$, *and* $\Xi = 0$, $\mathbf{d}\Xi \neq 0$ *on* $\mathscr{I}$.

(iii) There exists an embedding $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ *such that* $\varphi(\tilde{\mathcal{M}}) = \mathcal{M} \setminus \mathscr{I}$ *and*

$$\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}.$$

(iv) Each null geodesic of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *acquires two distinct endpoints on* $\mathscr{I}$.

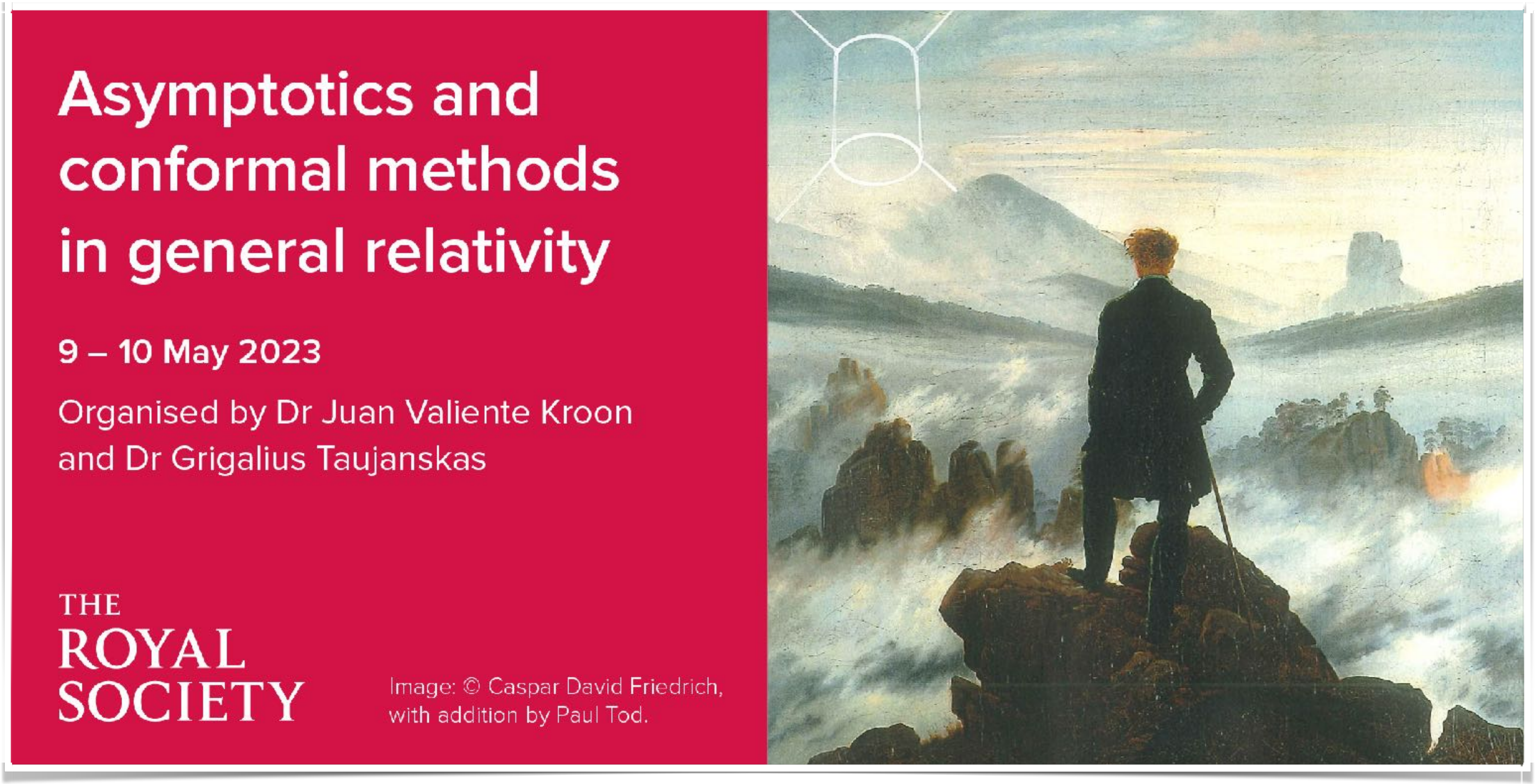


Motto:

Provide a geometric framework to study the asymptotics of the gravitational field!

$\mathscr{I}$ is 60!

Original ideas about the notion of asymptotic simplicity date to around 1963



Key aspect

The smoothness of $\mathscr{I}$

Smoothness at $\mathscr{I}^{\pm}$ ⟷ Decay of fields (peeling)

Corollary:

Restricted smoothness ⟷ Modified decay

Peeling

What do we mean exactly?

Smoothness is assumed here!

Theorem 1. *Let* $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ *denote a vacuum asymptotically simple spacetime with vanishing Cosmological constant. Then the components of the Weyl tensor with respect to a frame adapted to a foliation of outgoing light cones satisfy*

$$\tilde{\psi}_0 = O\left(\frac{1}{\tilde{r}^5}\right), \quad \tilde{\psi}_1 = O\left(\frac{1}{\tilde{r}^4}\right), \quad \tilde{\psi}_2 = O\left(\frac{1}{\tilde{r}^3}\right), \quad \tilde{\psi}_3 = O\left(\frac{1}{\tilde{r}^2}\right), \quad \tilde{\psi}_4 = O\left(\frac{1}{\tilde{r}}\right),$$

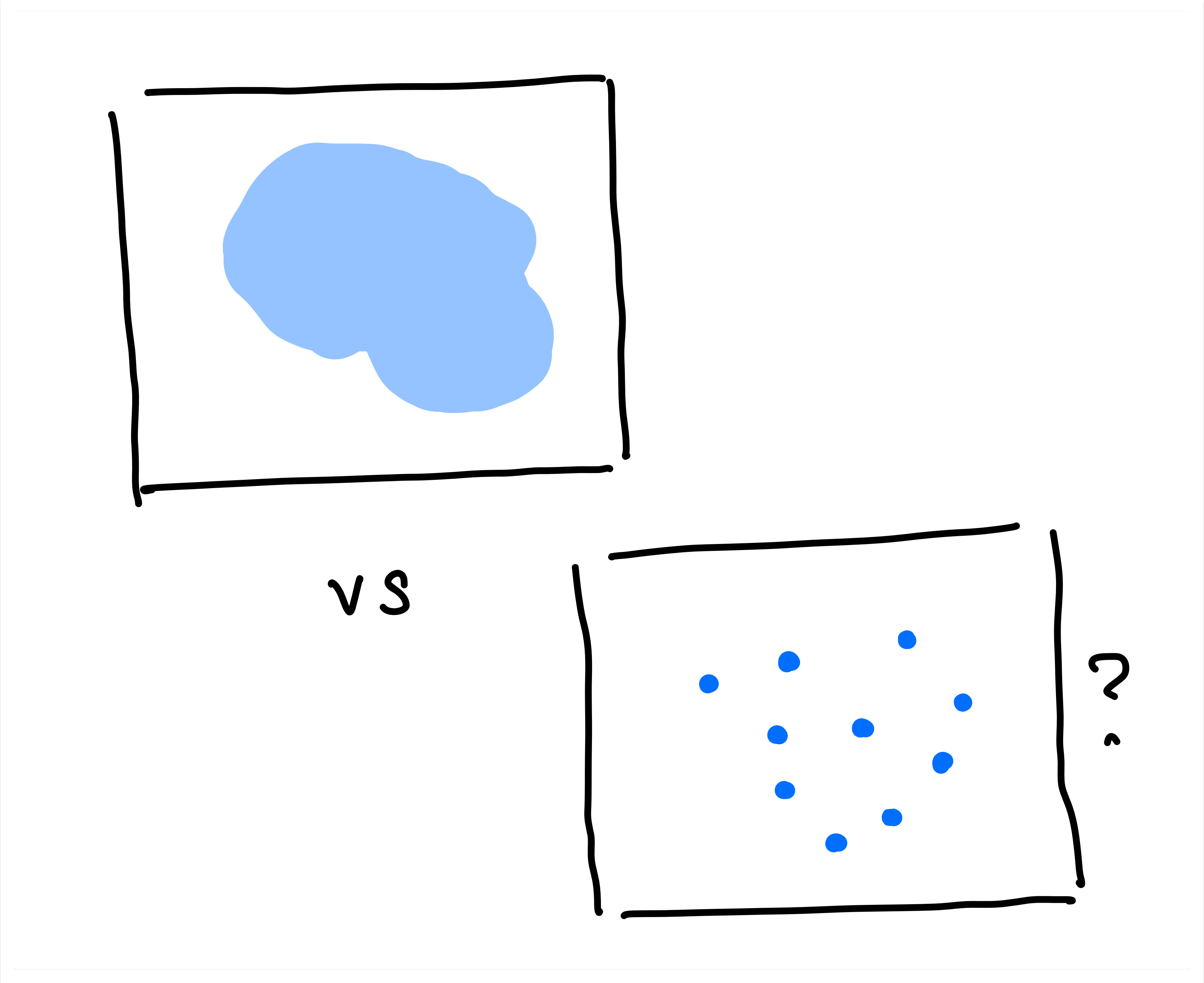
where $\tilde{r}$ *is a suitable parameter along the generators of the light cones.*

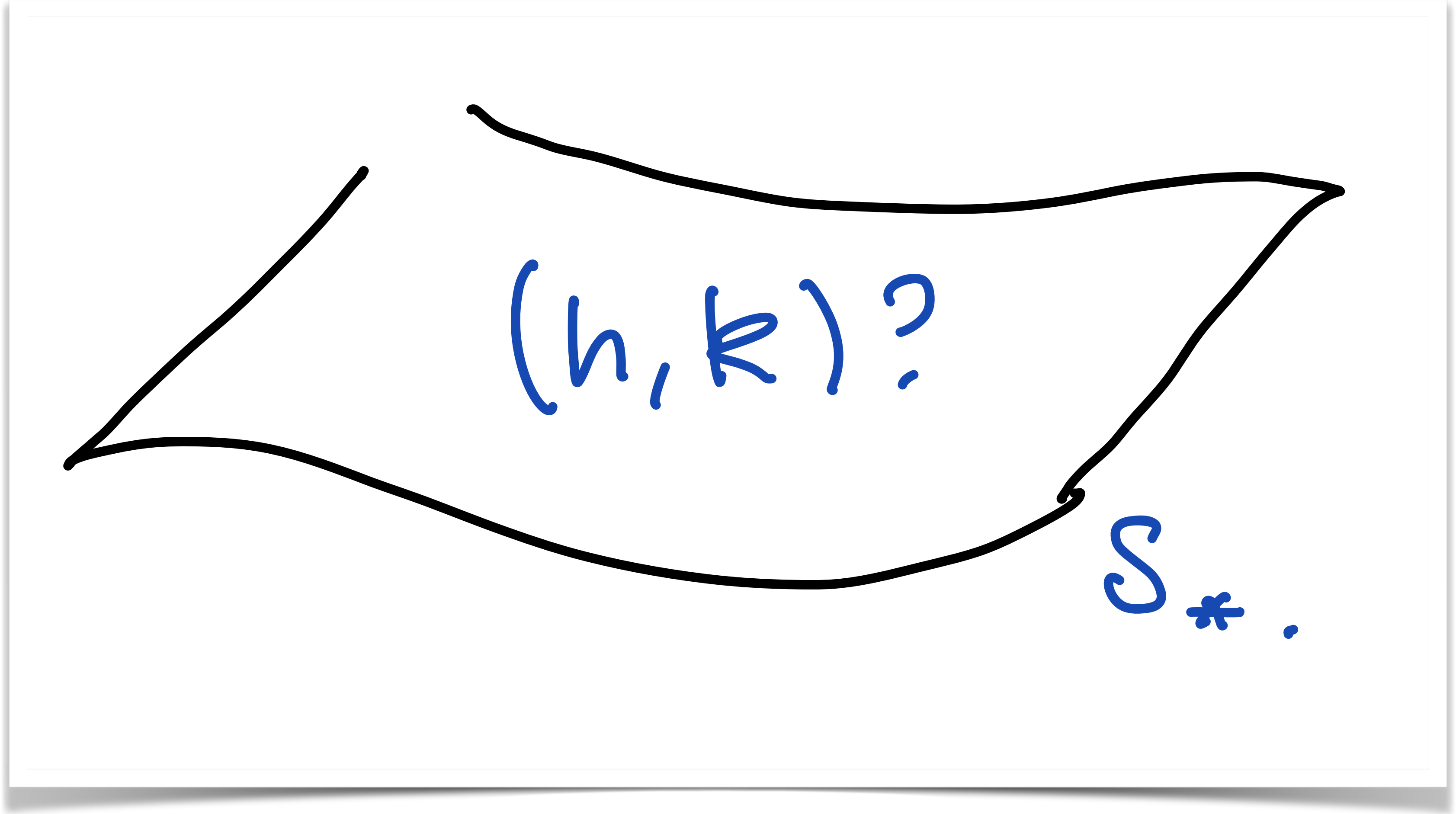
Penrose (1965)

Some natural questions

Genericity and the Cauchy problem

i. ***How large*** is the class of spacetimes with a smooth Penrose compactification?

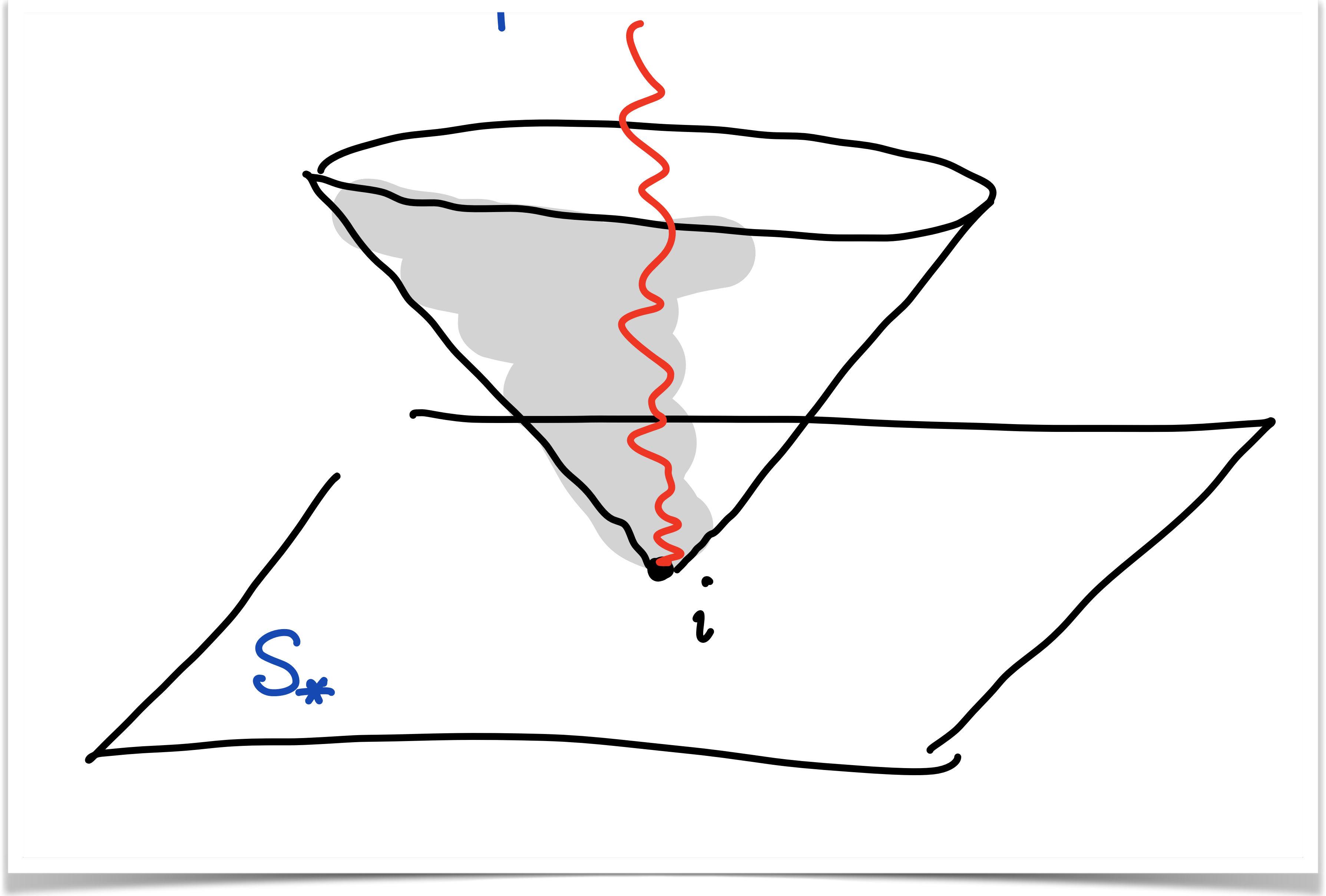




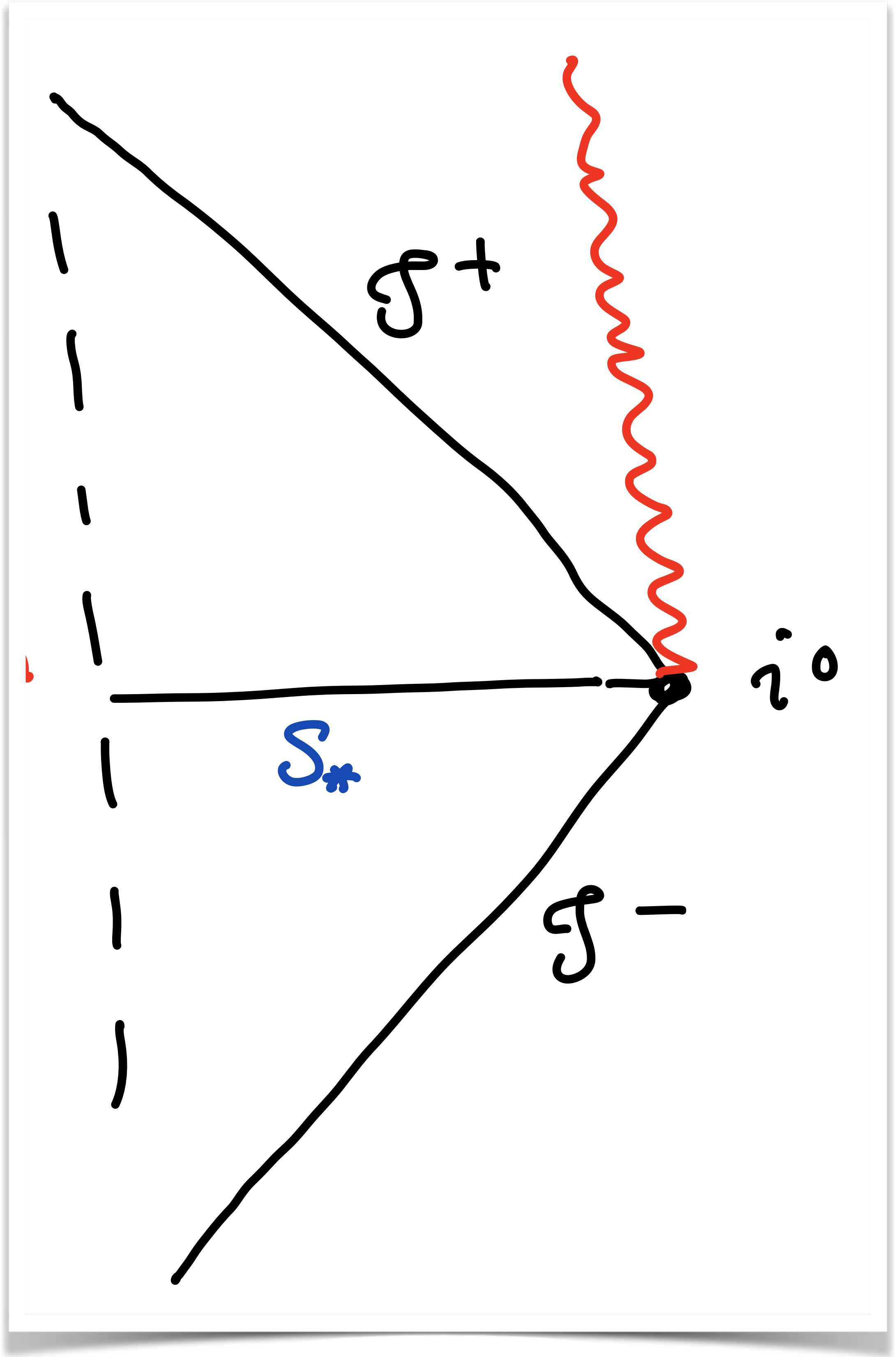
ii. How to construct the spacetime from, eg ***Cauchy initial data***? What extra conditions are required?

The *problem of spatial infinity*

The presence of mass produces a singularity of the conformal structure at i^0

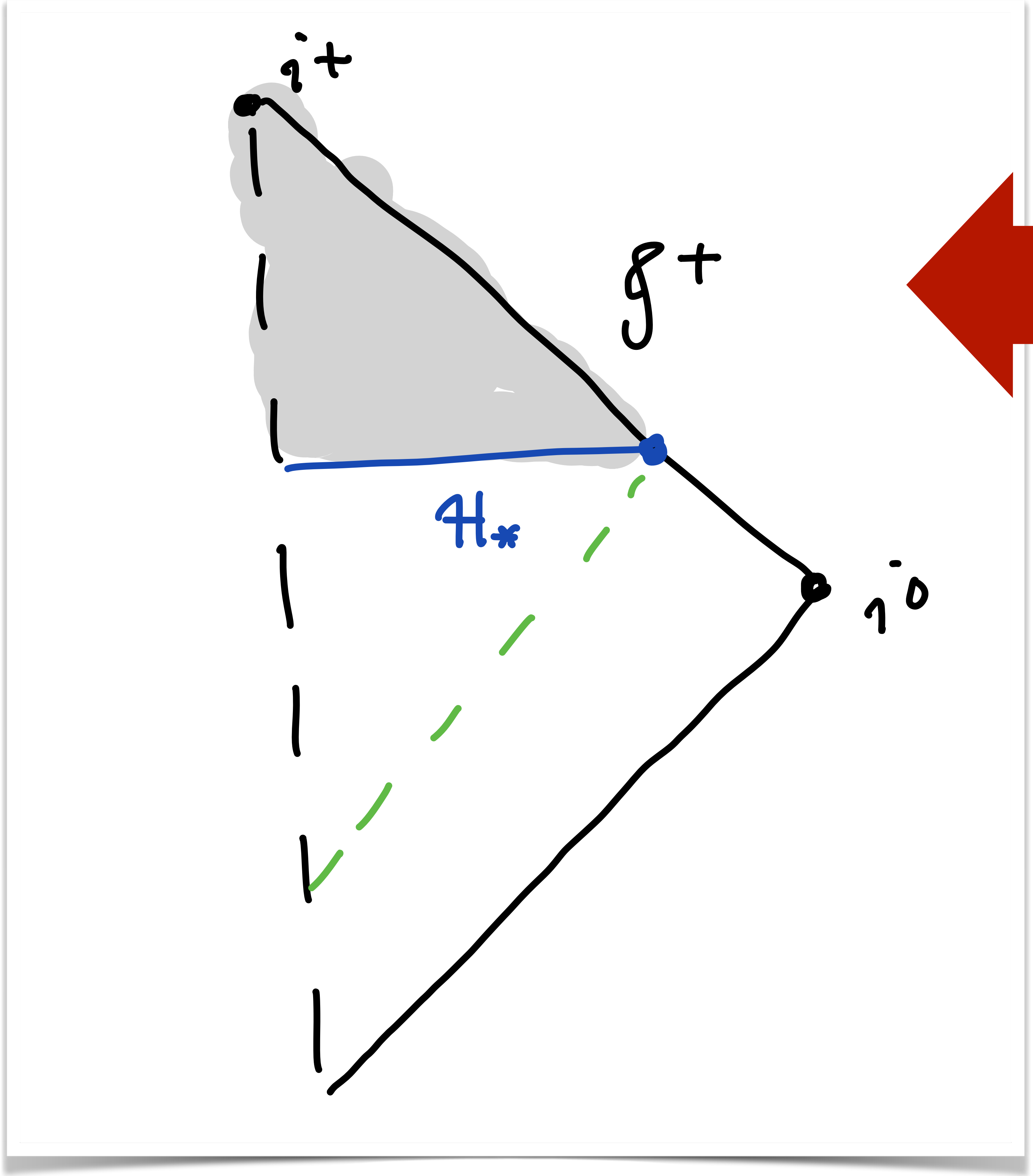


Penrose, 1965



Semiglobal stability of the Minkowski spacetime

Friedrich (1986)

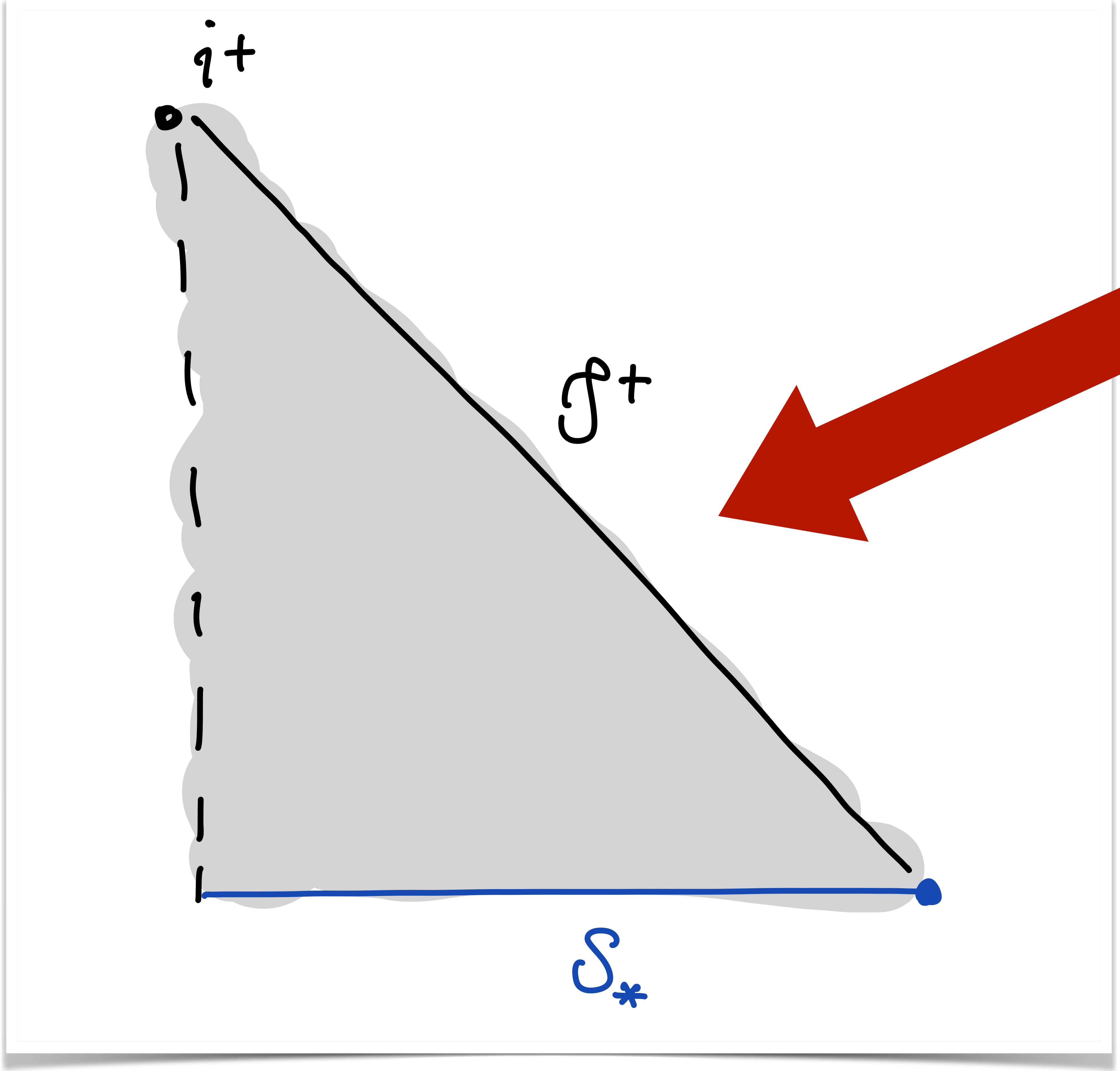


For suitable hyperboloidal data one can recover a smooth $\mathscr{I}^+$ on $D^+(\mathscr{H}_\star)$

What about Cauchy data?

Global non-linear stability of the Minkowski spacetime

D Christodoulou & S Klainerman (1990)

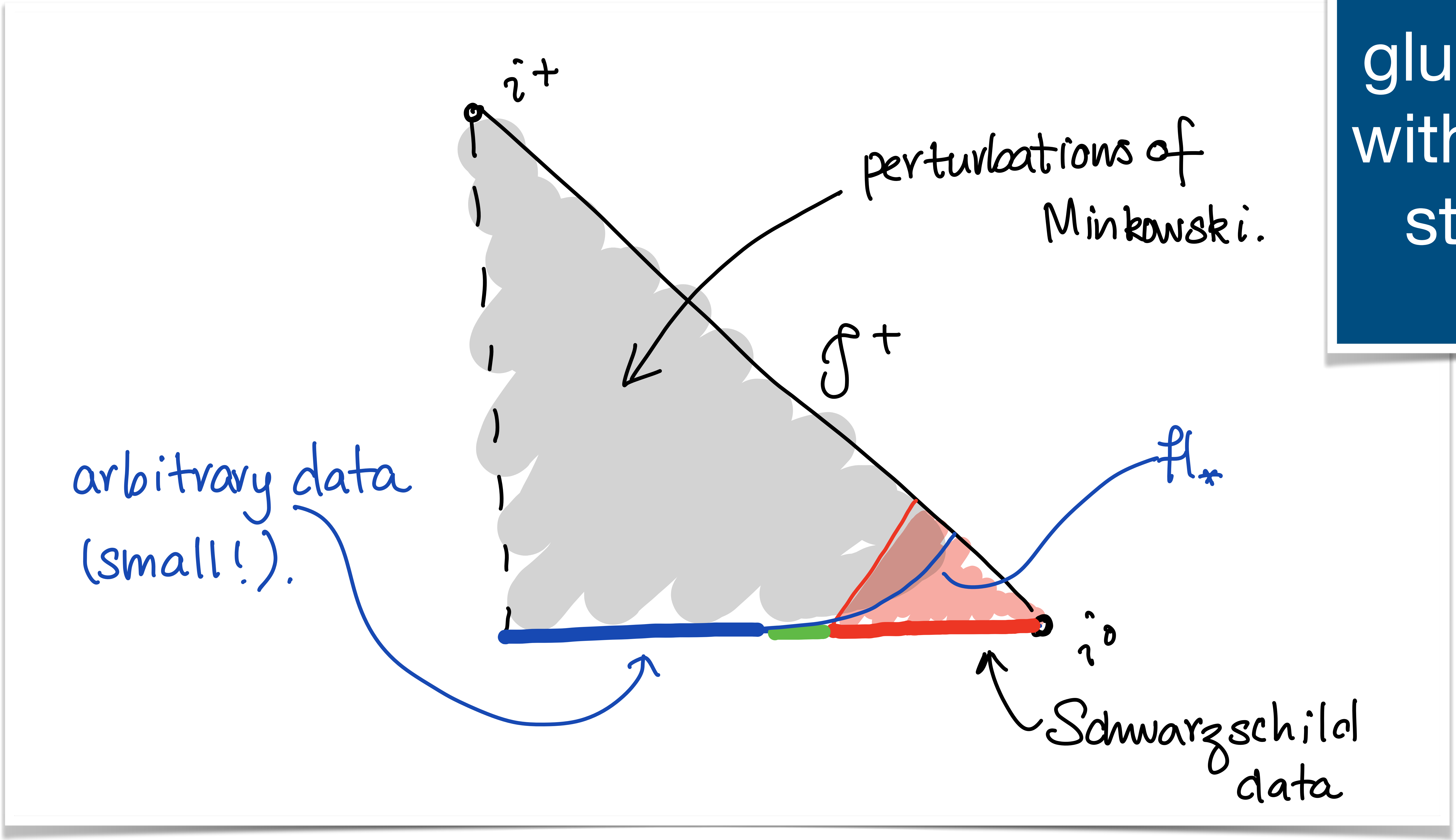


Cannot recover peeling!
Non-smooth $\mathscr{I}^+$!

Is this a technical problem or there is something more fundamental?

Gluing techniques

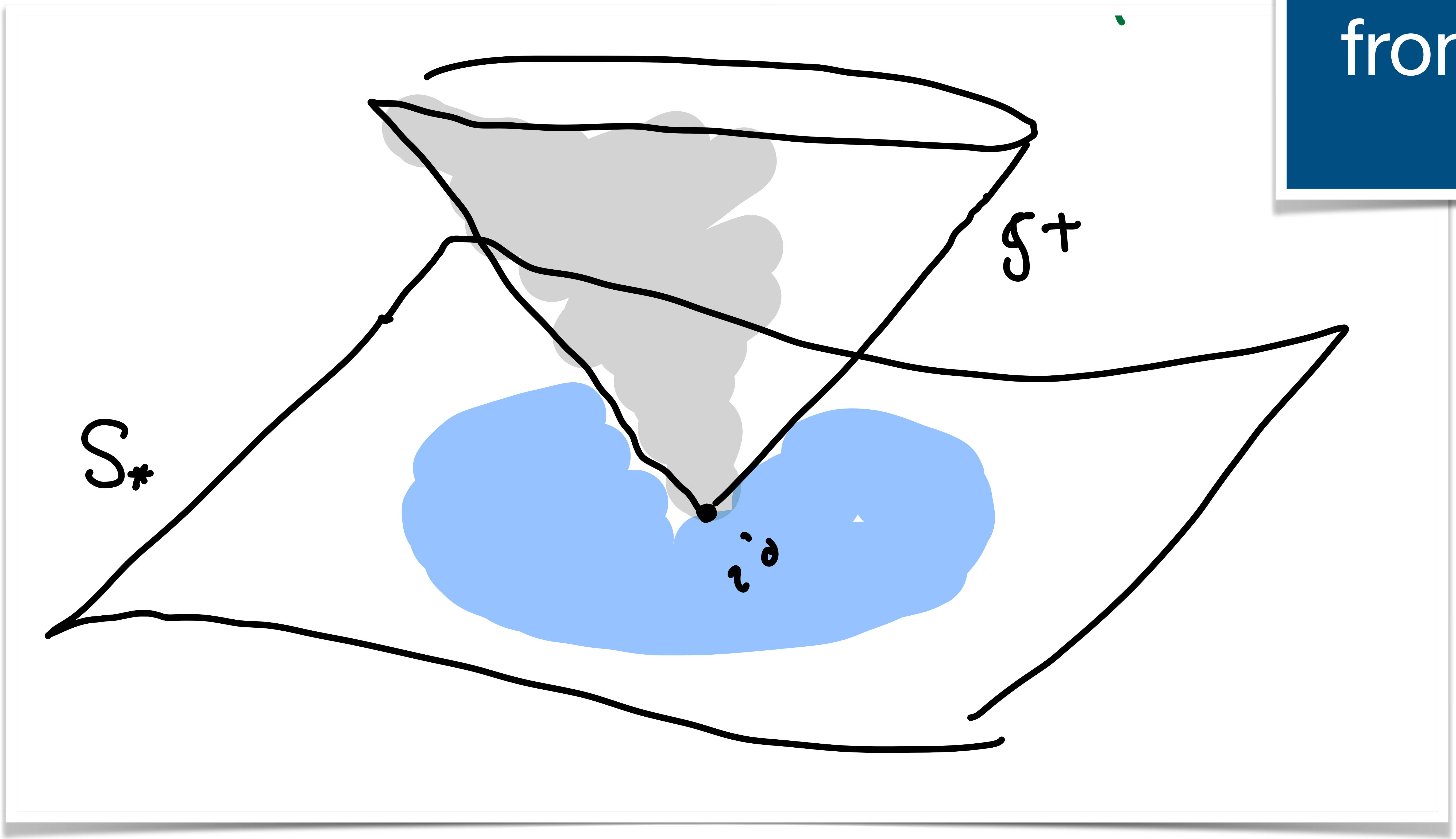
PT Chrusciel & E Delay (2005)



Use the Corvino-Schoen gluing techniques together with Friedrich's semi-global stability to construct AS spacetimes

Regular asymptotic initial value problem at spatial infinity

H Friedrich (1998)



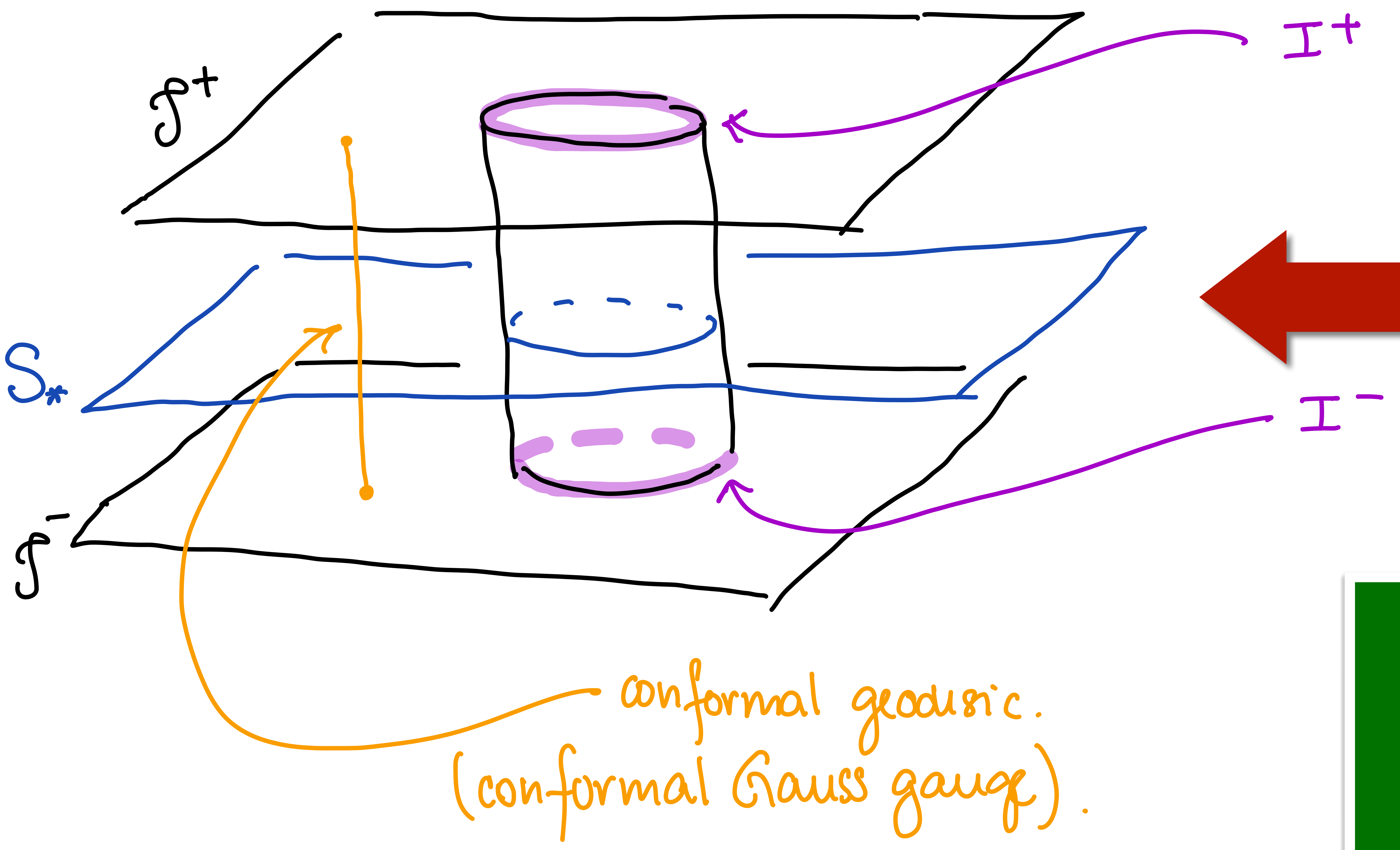
A detailed study of the structure of spatial infinity from the point of view of an IVP

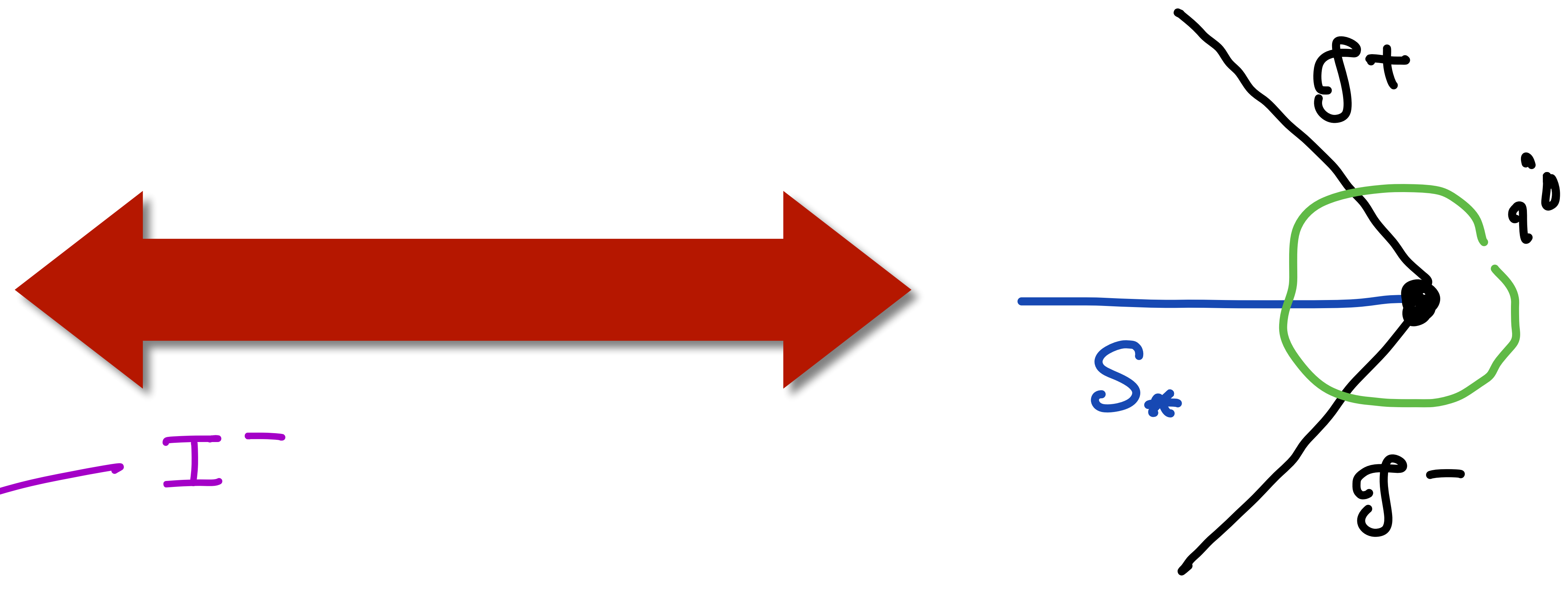
Regular

The equations and data are regular at spatial infinity

The cylinder at spatial infinity

Friedrich (1998) —see also Ashtekar & Hansen (1978), Beig & Schmidt (1984)





Relation to Ashtekar's framework for spatial infinity —M Magdy & JAVK, (2021)

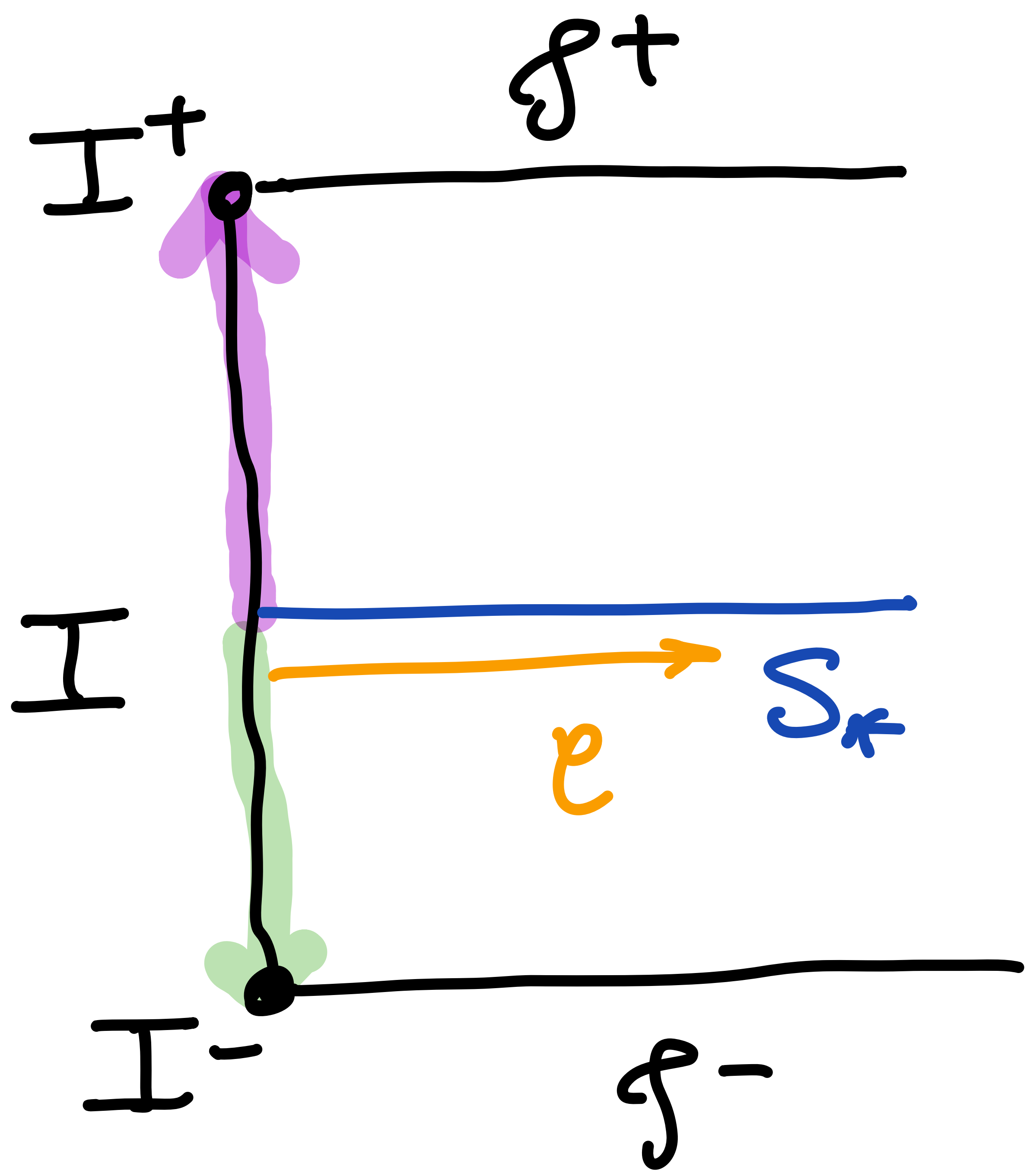
The cylinder at spatial infinity

The cylinder I is a total characteristic of the (conformal) Einstein field equations

All the evolution equations reduce to transport equations on the cylinder I

Data on $I_\star \Longleftrightarrow$ Solutions at $I^\pm$

Solution jets: $J[\phi^{(p)}] = \{(\partial_\rho^p \phi)\,|_I\}$



Obstructions to the smoothness of $\mathscr{I}$

Null infinity is generically non-smooth!

The regularity of the coefficients $\phi^{(p)}$ can be explicitly computed (modulo computational complexities)

Logarithmic divergences at $I^{\pm}$!!
—H. Friedrich (1998), JAVK (2004)

Data needs to be fine-tuned to obtain suitably regular solutions

Consequences of the non-smoothness of $\mathscr{I}$

Non-smooth solutions may not peel!

Time symmetric data!

Assumption 1. *The metric* $\boldsymbol{h}$ *satisfies the boundary conditions (11) with a conformal factor* $\Omega \in C^2(\mathcal{S}) \cap C^\infty(\mathcal{S} \setminus \{i\})$*. Moreover, it is analytic in a neighbourhood of* i *and there exists coordinates* $\underline{x} = (x^\alpha)$ *for which the components of* $\boldsymbol{h}$ *satisfy (12).*

$$\Omega(i) = 0, \qquad \mathbf{d}\Omega(i) = 0, \qquad \mathbf{Hess}\,\Omega(i) = -2\boldsymbol{h}(i), \tag{11}$$

$$h_{\alpha\beta} = -\delta_{\alpha\beta} + O(|x|^3). \tag{12}$$

Theorem 2. *Under assumption 2, given time symmetric initial data satisfying assumption 1, the F-expansions imply that*

$$\begin{aligned}
\tilde{\psi}_0 &= O(\tilde{r}^{-3}\ln\tilde{r}),\\
\tilde{\psi}_1 &= O(\tilde{r}^{-3}\ln\tilde{r}),\\
\tilde{\psi}_2 &= O(\tilde{r}^{-3}\ln\tilde{r}),\\
\tilde{\psi}_3 &= O(\tilde{r}^{-2}),\\
\tilde{\psi}_4 &= O(\tilde{r}^{-1}).
\end{aligned}$$

E Gasperín & JAVK (2017)

Some further examples

Regularity improves as one fine-tunes the data…

Theorem 3. *Under assumption 2, given time symmetric initial data satisfying assumption 1, the F-expansions are such that:*

(i) If

$$b_{ij}(i) = 0,$$

then

$$\tilde{\psi}_0 = O(\tilde{r}^{-4}\ln\tilde{r}),$$
$$\tilde{\psi}_1 = O(\tilde{r}^{-4}\ln\tilde{r}),$$
$$\tilde{\psi}_2 = O(\tilde{r}^{-3}),$$
$$\tilde{\psi}_3 = O(\tilde{r}^{-2}),$$
$$\tilde{\psi}_4 = O(\tilde{r}^{-1}).$$

(ii) If

$$b_{ij}(i) = 0, \qquad D_{\{k}b_{ij\}}(i) = 0,$$

then

$$\tilde{\psi}_0 = O(\tilde{r}^{-5}\ln\tilde{r}),$$
$$\tilde{\psi}_1 = O(\tilde{r}^{-4}),$$
$$\tilde{\psi}_2 = O(\tilde{r}^{-3}),$$
$$\tilde{\psi}_3 = O(\tilde{r}^{-2}),$$
$$\tilde{\psi}_4 = O(\tilde{r}^{-1}).$$

(iii) The classical peeling behaviour is obtained if

$$b_{ij}(i) = 0, \qquad D_{\{k}b_{ij\}}(i) = 0, \qquad D_{\{k}D_l b_{ij\}}(i) = 0.$$

The role of time independent solutions

Is stationarity near i^0 the only possibility?

Stationary spacetimes are as regular near spatial infinity in as one would expect —in particular, **the solutions do not have logarithmic singularities!**

Is there any type of rigidity implied by smoothness at spatial infinity?

Polyhomogeneous spacetimes

P Hintz & Vasy (2019)

Global non-linear stability of the Minkowski spacetime with polyhomogeneous expansions

Logarithms in the expansions

Not sharp but an important step forward! Relies on techniques of Melrose's school of microlocal analysis

Why care?

A framework to study spatial infinity

The conclusions from the analysis of i^0 are **generic** —i.e.independent from the set up of stability

Friedrich's cylinder at spatial infinity provides a framework for the study of asymptotic charges and other observables and **their relation to initial data**!

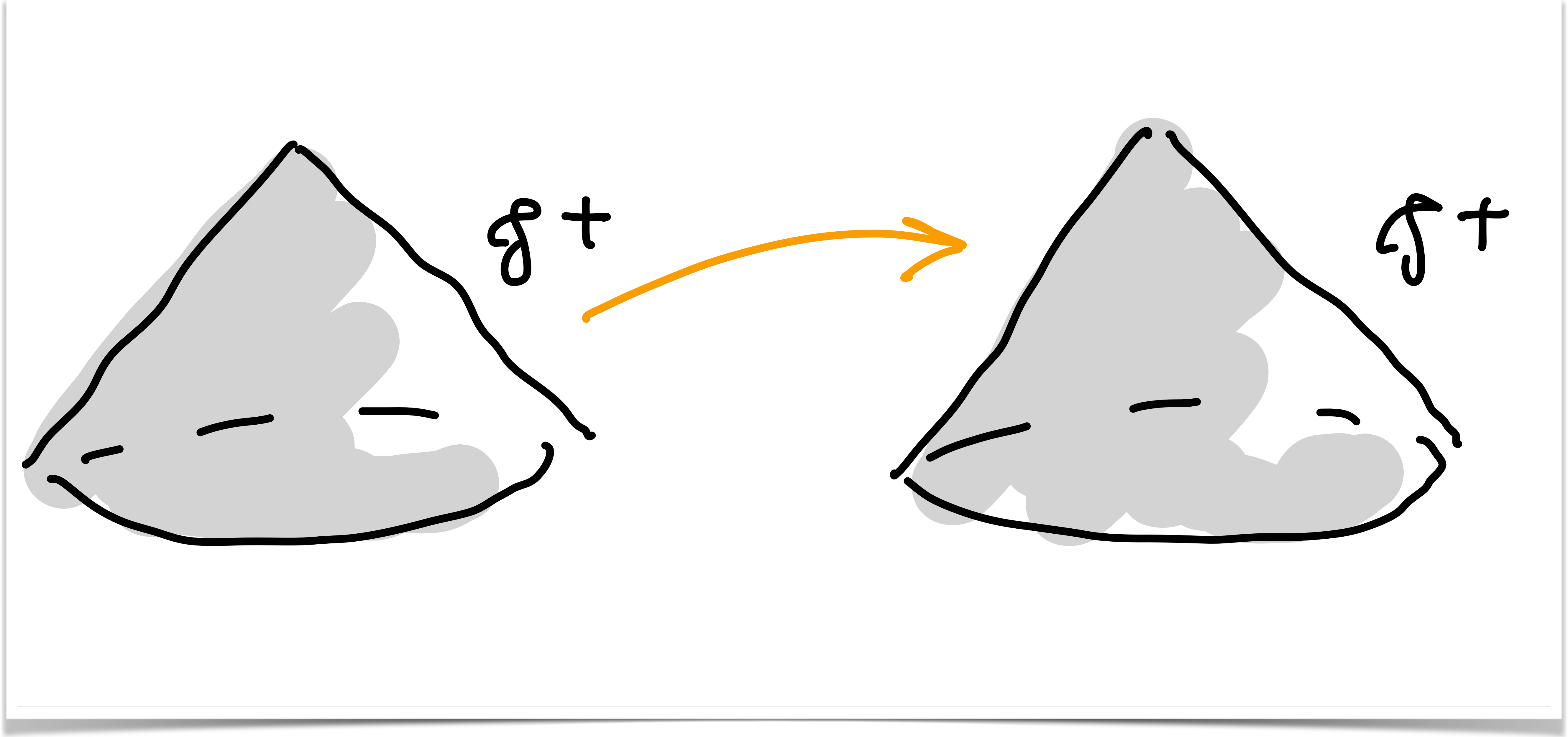
BMS charges and i^0

BMS charges

The symmetry group of $\mathscr{I}$ is the BMS (Bondi-Metzner-Sachs) group

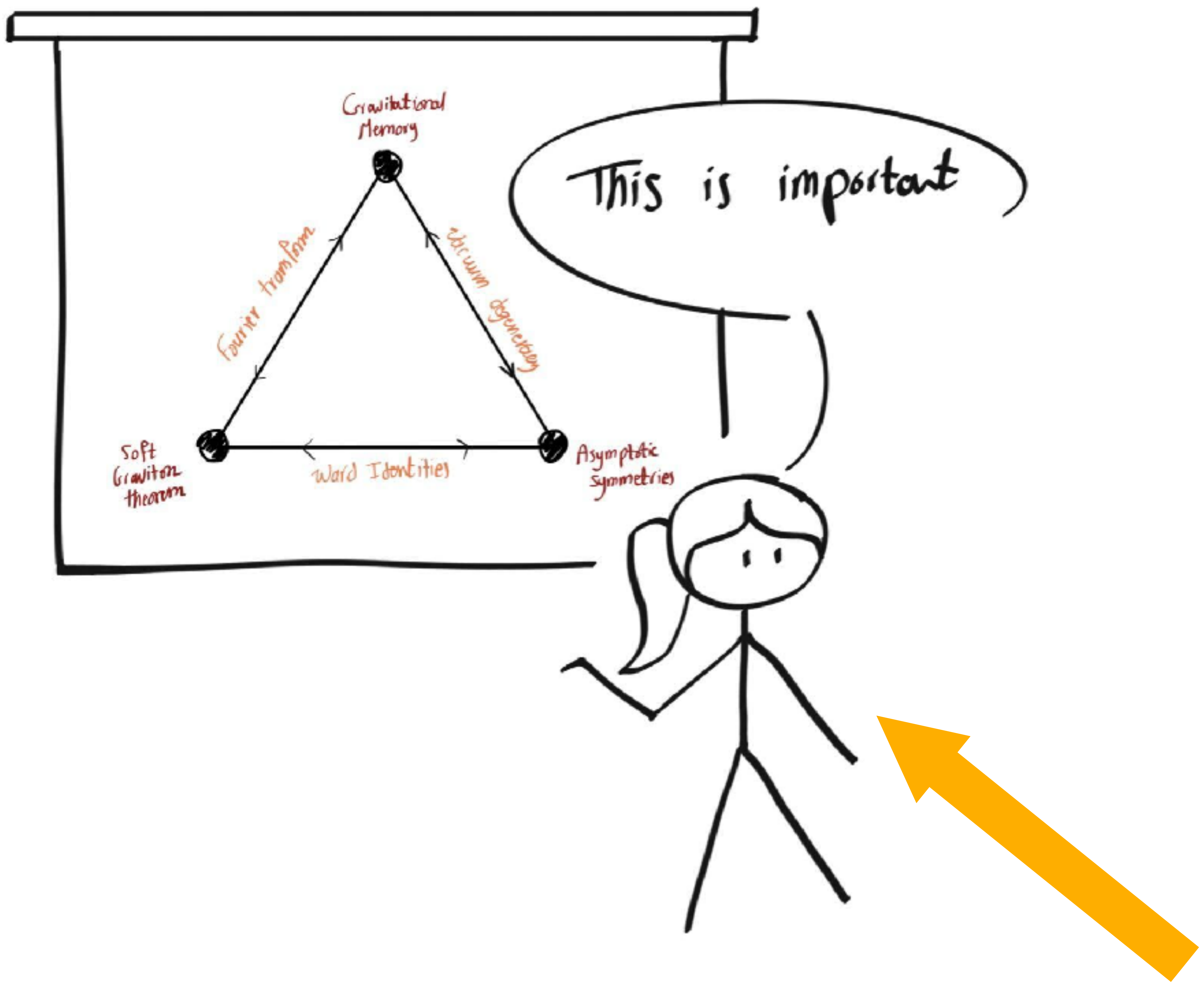
Asymptotic symmetries:

- Solutions to the asymptotic Killing equations
- Transformations of $\mathscr{I}$ preserving structure



Why we care?

The AS-GM-SGT triangle…



Credit: Mariem Magdy AM

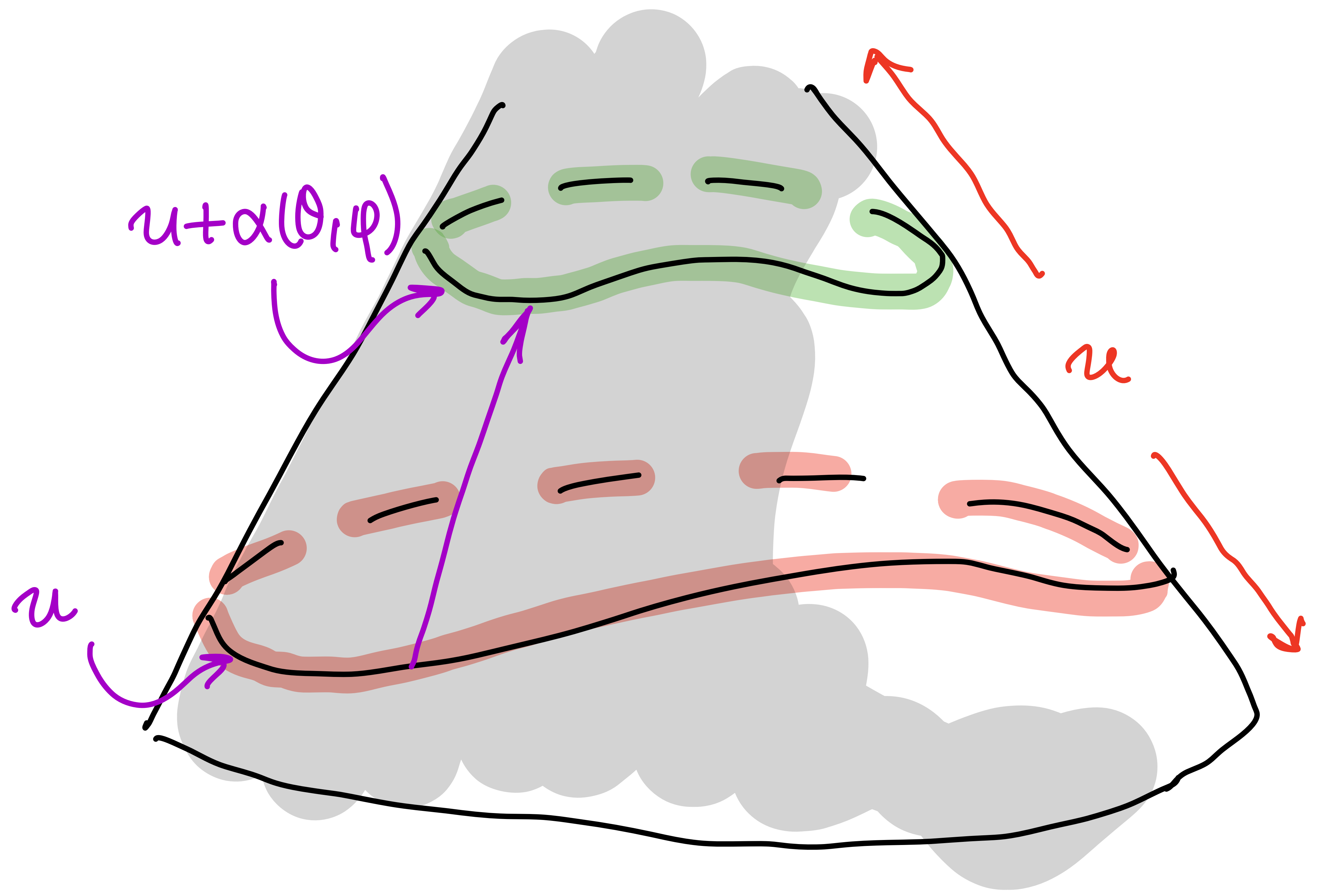
Supertranslations

Reparametrising the null generators of $\mathscr{I}$

- Of particular interest are supertranslations (reparametrisations of cuts of cuts of $\mathscr{I}$)

$$u \mapsto u + \alpha(\theta, \varphi)$$

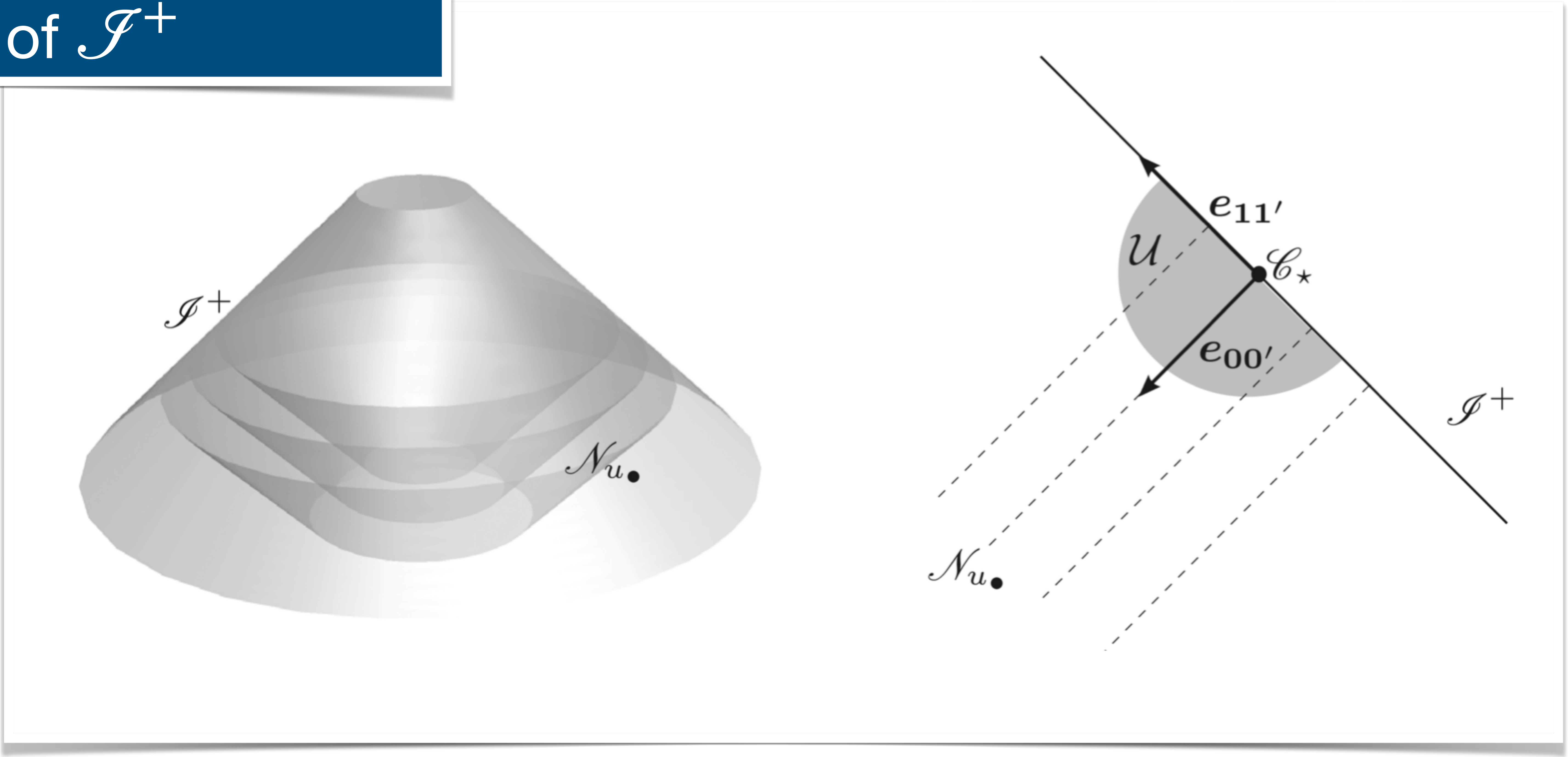
Smooth function on $\mathscr{C} \approx \mathbb{S}^2$



The Newman-Penrose gauge

ET Newman & R Penrose (1965), J Stewart (1984)

A choice of coordinates, frame and conformal scaling adapted to the geometry of $\mathscr{I}^+$



The NP gauge

A closer look…

Coordinates: $\bar{x} = (x^\mu) = (u, r, \theta, \varphi)$ (Bondi coordinates)

Frame: $\{\vec{l}', \vec{n}', \vec{m}', \vec{\bar{m}}'\} = \{\vec{e}'_{00'}, \vec{e}'_{11'}, \vec{e}'_{01'}, \vec{e}'_{10'}\}$

- $\vec{e}'_{00'}$ tangent to $\mathscr{I}^+$and $\nabla_{11'}\vec{e}'_{11'} = 0$
- $\vec{e}_{11'}(u) = 1$ on $\mathscr{I}^+$
- $\vec{e}'_{00'} = (\mathrm{d}u)^\sharp$

- The conformal freedom and residual freedom in the frame can be used to fix some components of the connection and Ricci tensor

The spin-2 equations

BMS charges for the spin-2 field

M Magdy AH & JAVK (JMP 2022)

$$\nabla^{A}{}_{A'}\phi_{ABCD} = 0, \qquad \phi_{ABCD} = \phi_{(ABCD)}$$

The BMS charge associated to a a super translation on a cut $\mathscr{C}$ of $\mathscr{I}^+$ is given by

$$\mathscr{Q} = 2\oint_{\mathscr{C}} \lambda\bar{\phi}_2 \mathrm{d}S$$

K Prabhu

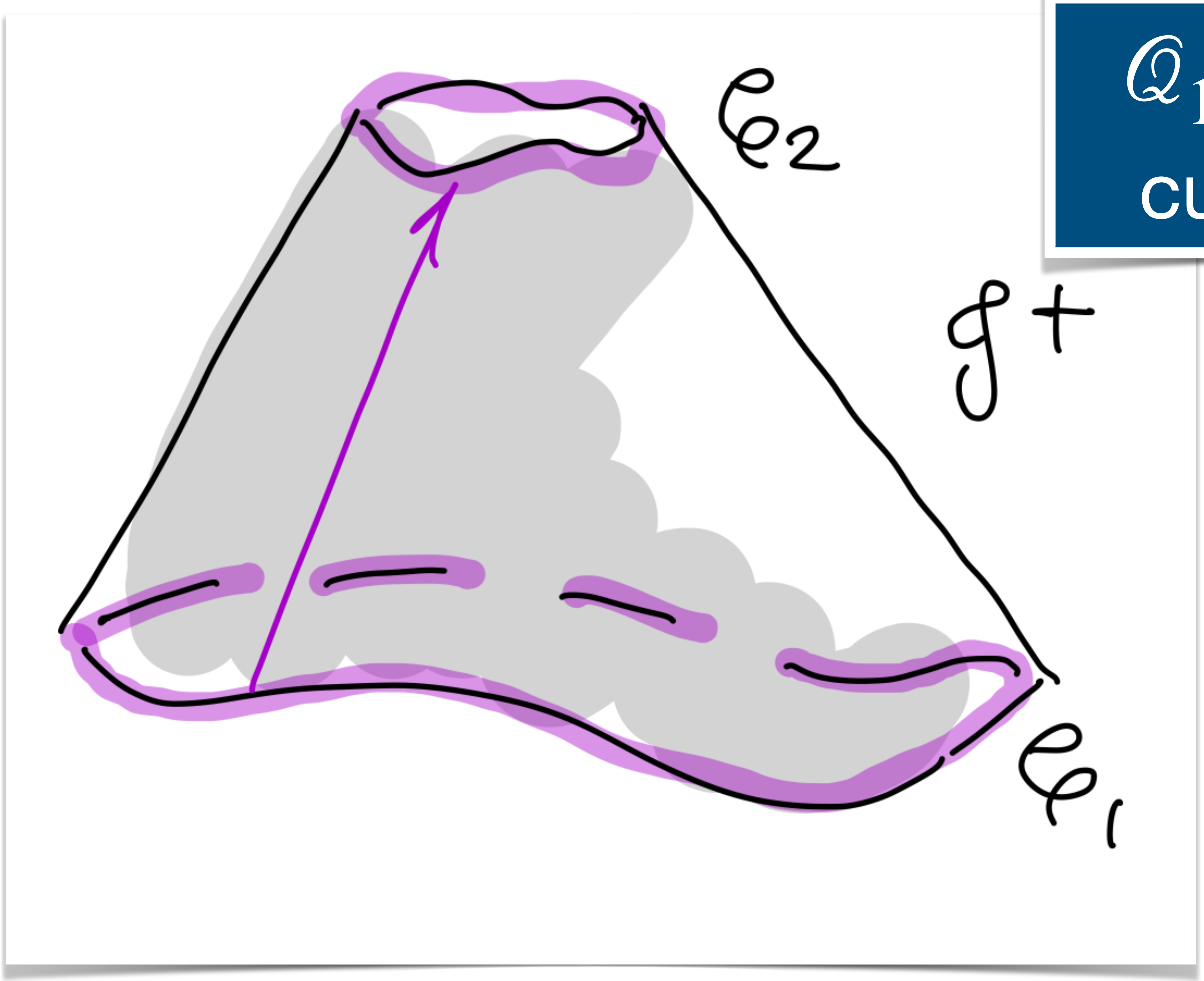
ϕ_2 the Coulomb field

λ a smooth function on $\mathbb{S}^2$

e.g. Y_{lm} (spherical harmonics)

Non-conservation of the charges

The value of the charges differs from cut to cut…



$\mathcal{Q}_1 \neq \mathcal{Q}_2$ for two cuts $\mathscr{C}_1$ and $\mathscr{C}_2$

A similar computation can be carried out on $\mathscr{I}^-$

The BMS charges at i^0

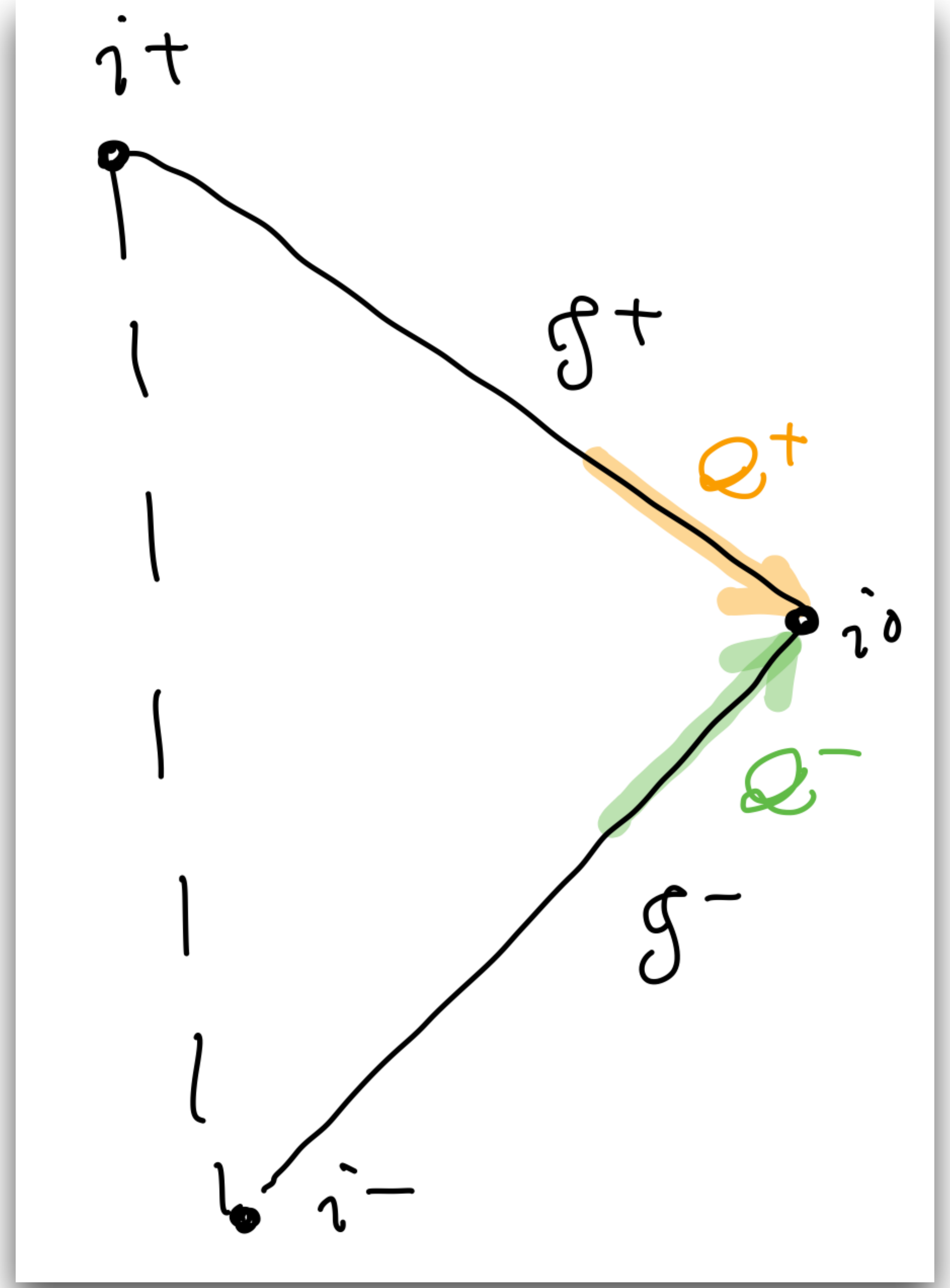
Taking things to the limit…

What happens if one considers the limit of $\mathscr{C}$ approaching the **critical sets** where $\mathscr{I}^{\pm}$ meets i^0?

Under which conditions are the limits well defined?

Are the charges on $\mathscr{I}^+$ and $\mathscr{I}^-$ related in some way?

Matching problem!



Stroeminger, Henneaux & Troessaert, Prabhu…

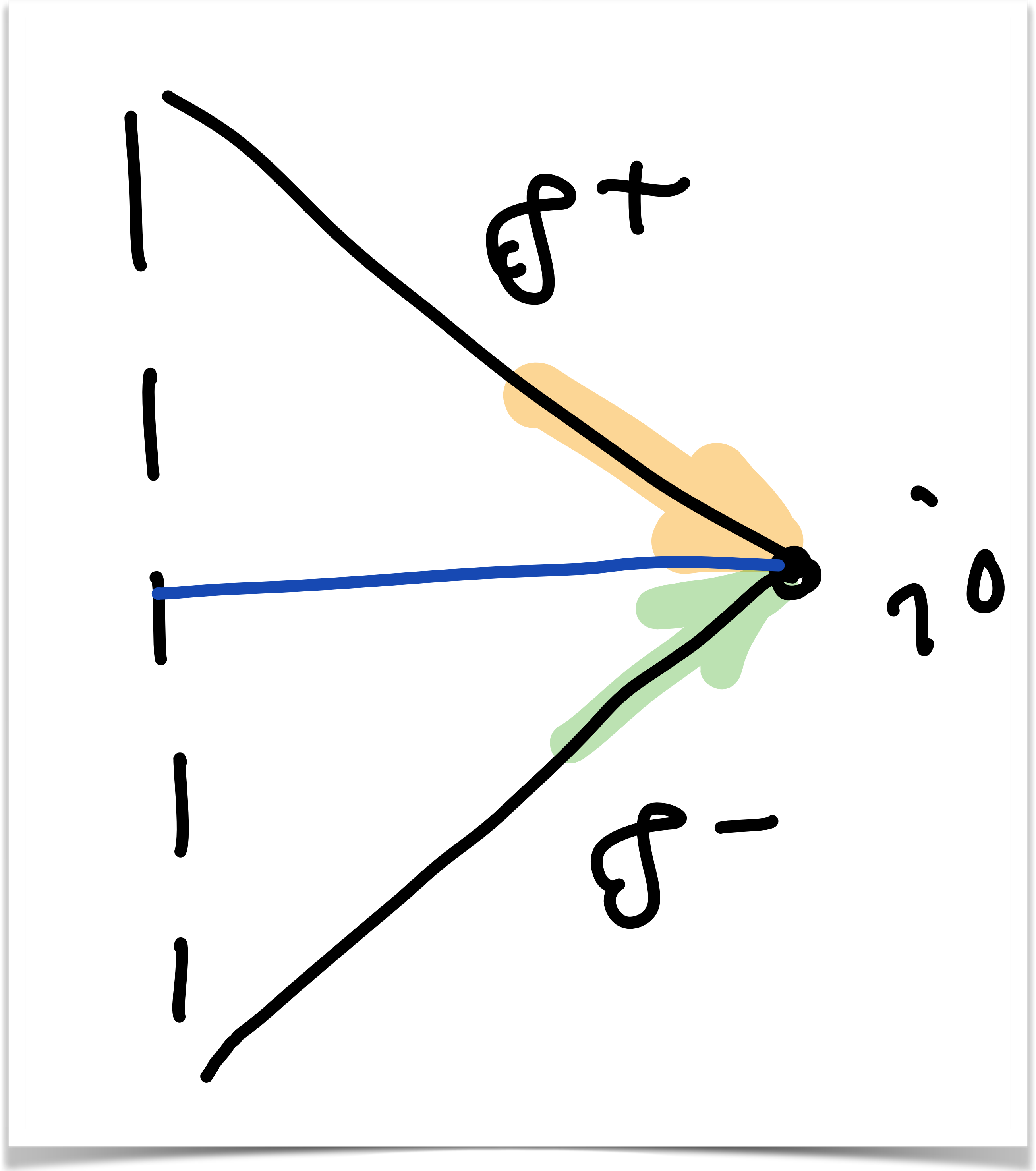
The initial value problem at spatial infinity

Study the matching problem for the BMS charges using an IVP…

Use Friedrich's representation of spatial infinity

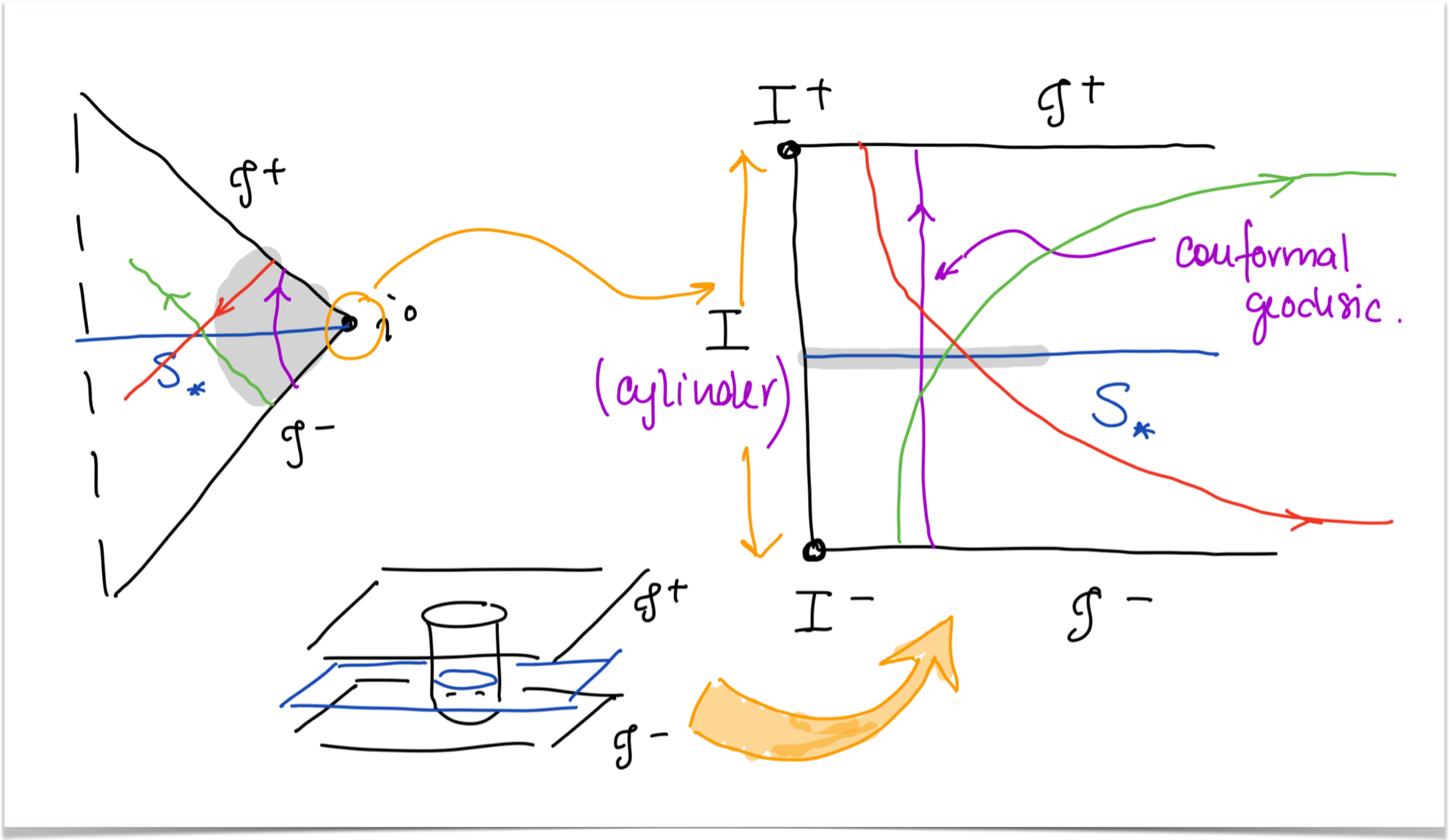
Advantage: assumptions controlled in terms of initial data

Write the charges in terms of **free data**



Friedrich's cylinder at spatial infinity

The geometric setup...



The F (Friedrich)-gauge

A conformal Gaussian system…

Based on a non-intersecting congruence of **conformal geodesics** in a neighbourhood of i^0

Coordinates: $(\tau, \rho, x^{\mathscr{A}})$

Frame: $\{\vec{e}_{AA'}\} = \{\vec{l}, \vec{n}, \vec{m}, \vec{\bar{m}}\}$

Well propagated along the conformal geodesics

In this gauge:

$$\mathscr{I}^+ = \{\tau = 1\}$$
$$\mathscr{I}^- = \{\tau = -1\}$$

Relating the NP and F gauges

Friedrich & Kánnár (2000)

Proposition 1. *The NP-gauge frame at $\mathscr{I}^+$ and F-gauge frame in the Minkowski spacetime are related via*

$$\boldsymbol{e}'_{\boldsymbol{AA'}} = \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}} \bar{\Lambda}^{\boldsymbol{B'}}{}_{\boldsymbol{A'}} \boldsymbol{e}_{\boldsymbol{BB'}}, \tag{3}$$

and

$$\Lambda^{\mathbf{1}}{}_{\mathbf{0}} = \frac{2e^{i\omega}}{\sqrt{\rho}(1+\tau)}, \qquad \Lambda^{\mathbf{0}}{}_{\mathbf{1}} = \frac{e^{-i\omega}\sqrt{\rho}(1+\tau)}{2},$$
$$\Lambda^{\mathbf{1}}{}_{\mathbf{1}} = \Lambda^{\mathbf{0}}{}_{\mathbf{0}} = 0, \tag{4}$$

where ω is an arbitrary real number that encodes the spin rotation of the frames on $\mathbb{S}^2$. For the NP-gauge frame at $\mathscr{I}^-$, the roles of the vectors $\mathbf{e}'_{\mathbf{00'}}$ and $\mathbf{e}'_{\mathbf{11'}}$ are interchanged, and NP-gauge frame is related to the F-gauge by equation (3) with $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$ given by

$$\Lambda^{\mathbf{1}}{}_{\mathbf{0}} = \frac{e^{-i\omega}\sqrt{\rho}(1-\tau)}{2}, \qquad \Lambda^{\mathbf{0}}{}_{\mathbf{1}} = \frac{2e^{i\omega}}{\sqrt{\rho}(1-\tau)},$$
$$\Lambda^{\mathbf{1}}{}_{\mathbf{1}} = \Lambda^{\mathbf{0}}{}_{\mathbf{0}} = 0. \tag{5}$$

The BMS charges at $\mathscr{I}^{\pm}$

Assumptions…

Key assumption: near $\mathscr{I}$ one has a solution of the form

$$\phi_2 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\tau) Y_{\ell m} + o(\rho)$$

Boosted data! Usually one has

$$a Y_{00} + o(\rho)$$

Existence of solutions of this form can be established using certain type of estimates (G Taujanskas & JAVK, 2023)

The subheading terms can be controlled

The cylinder at spatial infinity as a total characteristic

M Magdy AM & JAVK (2022)

The coefficients $a_{\ell m}(\tau)$ can be explicitly computed from transport equations on I. For example:

$$(1-\tau^2)\ddot{a}_{\ell m} - 2\tau\dot{a}_{\ell m} + \ell(\ell+1)a_{\ell m} = 0$$

The general solution is:

$$a_{\ell m}(\tau) = \mathfrak{a}_{\ell m}P_\ell(\tau) + \mathfrak{b}_{\ell m}Q_\ell(\tau)$$

Legendre function of the second kind

Legendre polynomial

$$Q_\ell(\tau) = \mathfrak{c}_\ell \ln(1 \pm \tau) + O(1)$$

Regularity at $\mathscr{I}^{\pm}$

One needs to fine-tune the data…

Proposition. The regular solutions at $\mathscr{I}^{\pm}$ are characterised by the conditions:

- $a_{\ell m}(0) = 0$ for ℓ odd
- $\dot{a}_{\ell m}(0) = 0$ for ℓ even

Gauss constraint:
$D^{AB}\phi_{ABCD} = 0$

These conditions can be characterised in terms of free data for the spin-2 field

$$\phi_{ABCD} = (\mathfrak{G}\psi)_{ABCD}$$

One can find a ψ_{ABCD} (free data) satisfying the regularity conditions

Third order operator, Andersson, Bäckdahl & Joudioux (2014)

The BMS charges in terms of the data

The limits are generically not well-defined…

If the solutions are well defined at $I^{\pm}$ one finds that:

- $\mathcal{Q}\big|_{I^+} = -2\bar{a}_{\ell m}(1)$
- $\mathcal{Q}\big|_{I^-} = -2\bar{a}_{\ell m}(-1)$

Moral take away: the limits are generically not well-defined unless one fine-tunes the data!

Identifying the charges at $\mathscr{I}^+$ and $\mathscr{I}^-$

No need of the antipodal identification...

When the charges are well-defined at $I^{\pm}$ one has that:

- $\mathcal{Q}^+ = \mathcal{Q}^-$ for ℓ even
- $\mathcal{Q}^+ = -\,\mathcal{Q}^-$ for ℓ odd

The antipodal matching is, in fact, a regularity condition!

The BMS charges in GR

(M Magdy AM, K Prabhu & JAVK, to appear in JMP)

The BMS charges for the Weyl tensor

In full non-linear GR corrections appear…

In this case the charges are given by:

$$\mathcal{Q} = \oint_{\mathscr{C}} \lambda\left(\phi_2 + \frac{1}{2}\sigma^{ab}N_{ab}\right)\mathrm{d}S,$$

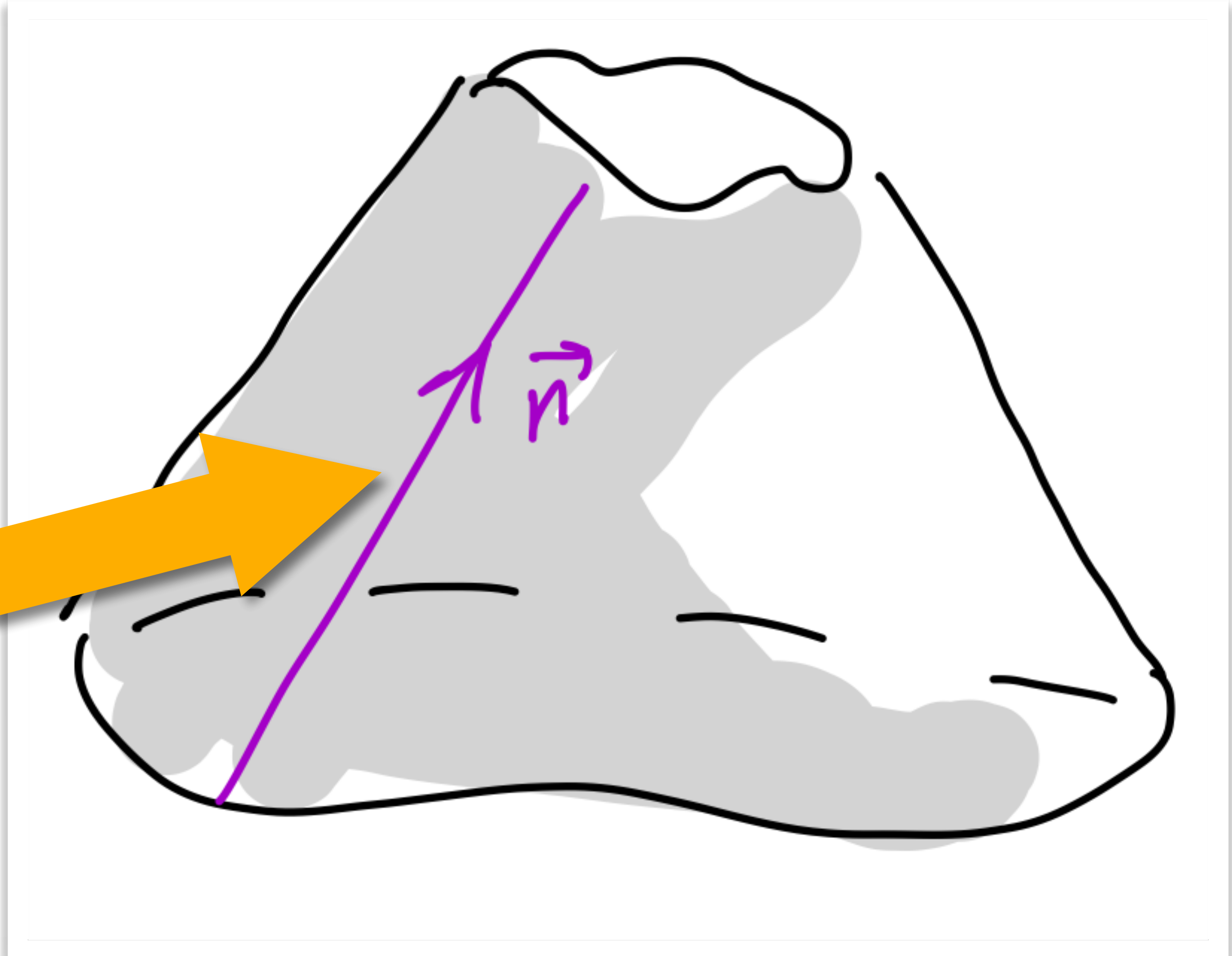
with:

- σ_{ab} the shear tensor on $\mathscr{I}^+$
- $N_{ab} \equiv 2(\pounds_n - \Phi)\sigma_{ab}$
- $\Phi \equiv \frac{1}{4}\left(\nabla_a n^a\right)|_{\mathscr{I}^+}$

ϕ_2 the Coulomb component of the rescaled Weyl tensor $\phi_{ABCD} = \Theta^{-1}\Psi_{ABCD}$

λ a smooth function over $\mathbb{S}^2$

n^a null geodesic generator of $\mathscr{I}^+$



Choosing the initial data

L-H Huang CQG 27, 245002 (2010)

Proposition 2. For any $\alpha, \beta \in C^2(\mathbb{S}^2)$ and $q \geq 1$, there exists a vacuum initial data set $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{\pi}})$ where the components of $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{\pi}}$ with respect to the standard Euclidean coordinate chart $\{x^\alpha\}$ have the following asymptotics:

$$\tilde{h}_{\alpha\beta} = -\left(1 + \frac{A}{r}\right)\delta_{\alpha\beta} - \frac{\alpha}{r}\left(\frac{x_\alpha x_\beta}{r^2} - \frac{1}{2}\delta_{\alpha\beta}\right) + O_2(r^{-1-q}),$$

$$\tilde{\pi}_{\alpha\beta} = \frac{\beta}{r^2}\frac{x_\alpha x_\beta}{r^2} + \frac{1}{r^3}\left(-B_\alpha x_\beta - B_\beta x_\alpha + (B^\gamma x_\gamma)\delta_{\alpha\beta}\right) + O_1(r^{-2-q}),$$

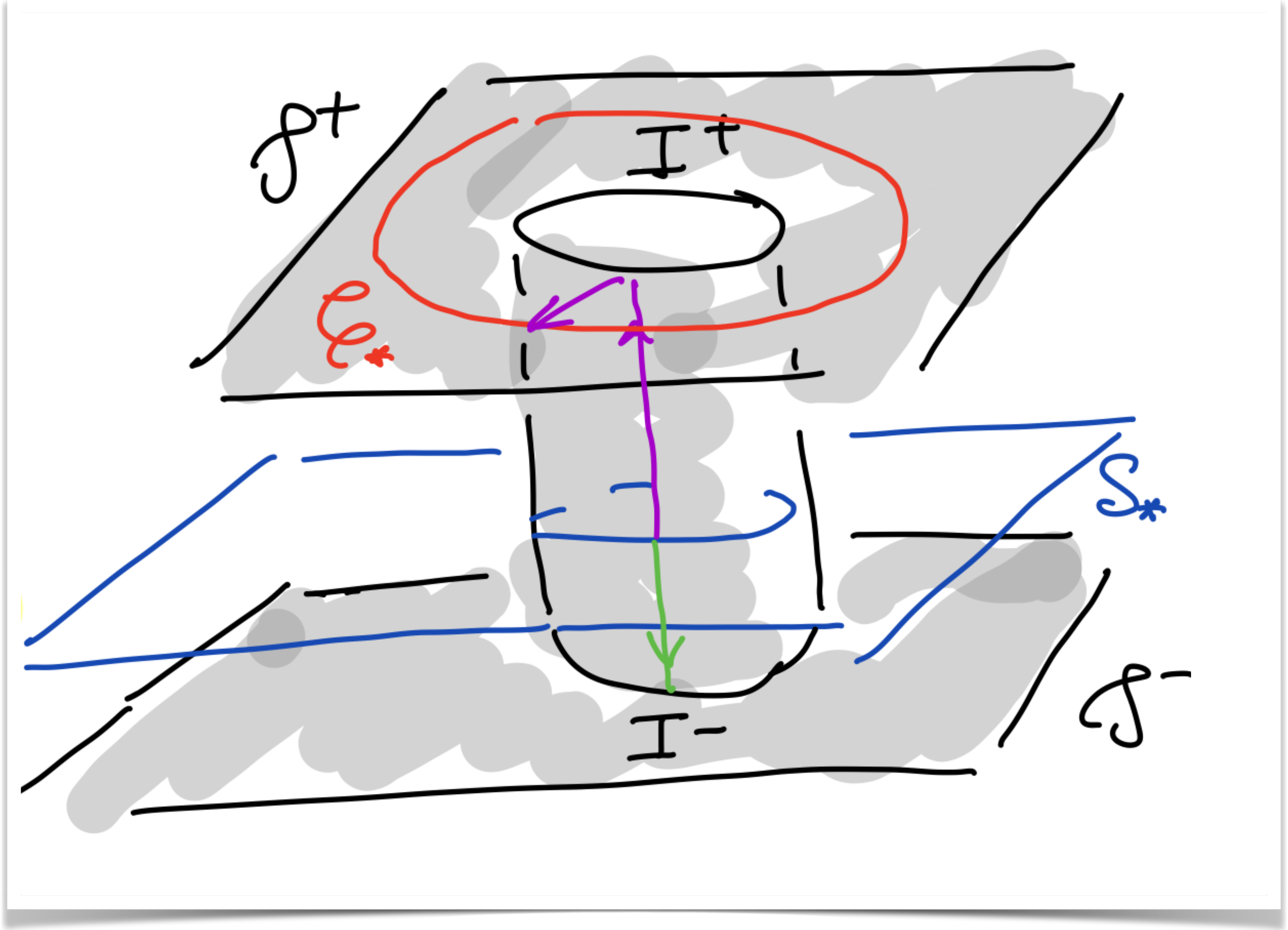
where A, $\{B_\alpha\}_{\alpha=1}^3$ are some constants, $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and $\tilde{\boldsymbol{\pi}}$ is the momentum tensor, related to $\tilde{\boldsymbol{K}}$ by

$$\tilde{\pi}_{ij} = \tilde{K}_{ij} - \tilde{K}\tilde{h}_{ij}. \tag{58}$$

The proof makes use of gluing techniques!

Computation of asymptotic expansions

Use again the total characteristic at spatial infinity...



For the above class of initial data one can make use of the properties of the cylinder I to compute asymptotic expansions of all the relevant fields:

$$\phi_{ABCD}, \quad \sigma_{ab}, \quad N_{ab}$$
$$\Lambda^A{}_B, \quad \vartheta$$

Give the transformation between frames

Caveat: the expansions are formal! One needs to adapt the methods of linear fields to GR or adapt the analysis of Hintz & Vasy.

Structure of the asymptotic expansions

GR behaves like spin-2 field…

The leading behaviour of ϕ_2 is given by

$$\phi_2 = \sum_{\ell=0}^{\infty} \sum_{\ell=-m}^{m} a_{\ell m}(\tau) Y_{\ell m} + O(\rho),$$

with, again,

$$a_{\ell m}(\tau) = \mathfrak{a}_{\ell m} P_\ell(\tau) + \mathfrak{b}_{\ell m} Q_\ell(\tau)$$

Logarithmic divergences!

Crucially, one has that

$$\sigma_{ab}|_{\mathscr{I}^\pm} \to 0$$

as one approaches $I^\pm$

The structure of the charges at $I^\pm$ is formally the same as for the spin-2 field!

Take away: the regularity of the solutions is controlled by conditions on the multipolar structure of α

Regularity of the solutions

The charges are, generically, not well defined…

Regular solutions are obtained if the odd parity harmonics (ℓ odd) in α vanish!

Only BMS super translation charges with ℓ even have non-trivial information!

Identifying the BMS charges at $\mathcal{I}^+$ and $\mathcal{I}^-$

The role of the initial data…

The BMS charges at $\mathcal{I}^\pm$ (when defined) are given in terms of the multipolar structure of α

This establishes the identification between $\mathcal{Q}^+$ and $\mathcal{Q}^-$

No antipodal map required for this!
Only sufficient regularity for the charges to be well defined…

Conclusions & outlook

Conclusions

Key take away messages…

- Friedrich's representation of spatial infinity can be used **to understand the assumptions behind asymptotic**

- Some of the standard assumptions are **non-generic**!

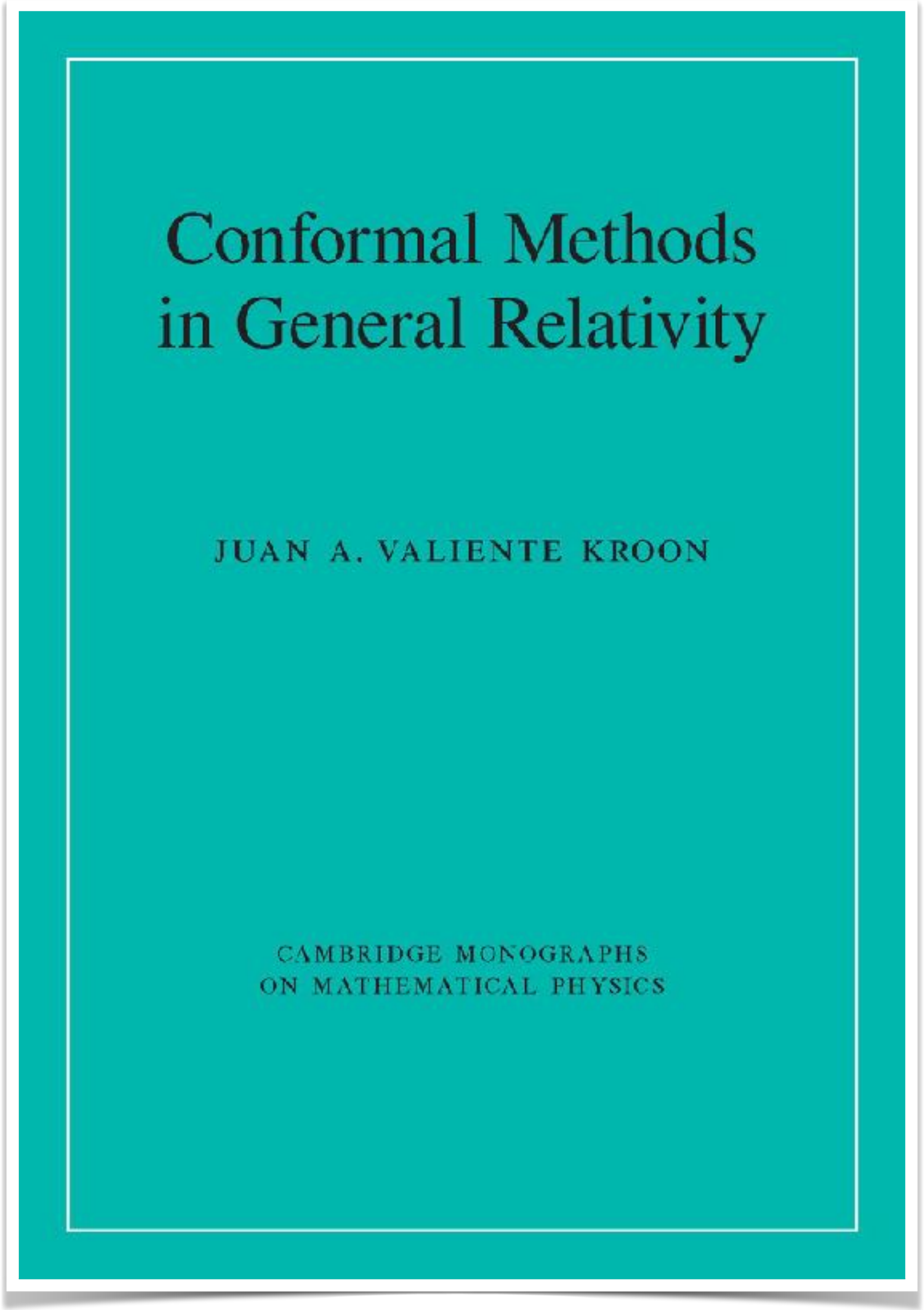
- Assumptions on free Cauchy data are ok!

Outlook

What lies ahead?

Wrap up H. Friedrich's programme with **rigorous statements** on the relation of asymptotic expansions and solutions to the Einstein field equations

Want to know more?



Thank you for your attention!