Soliton resolution on a wormhole

Piotr Bizoń

Jagiellonian University

Dedicated to the memory of Bernd Schmidt

10 November 2023

Introduction

- Given an evolution equation $\frac{du}{dt} = A(u)$ with initial data $u(0)$ we want to understand what happens as $t \to \infty$
- The set of possible endstates is 'smaller' than the set of initial data $\Rightarrow$ dynamical asymptotic simplification
- Soliton resolution conjecture: for generic global-in-time solutions of nonlinear dispersive wave equations on *unbounded* domains

  $$u(t) \sim \sum_i Q_i + \text{radiation} \qquad \text{for } t \to \infty$$

  where the 'solitons' Q_i are asymptotically decoupled.
- The conjecture has been proved for some integrable equations and recently for radial critical wave equations (Jendrej-Lawrie, Merle et al.). There is also extensive numerical and experimental evidence.
- This talk: soliton resolution in a simple geometric setting employing hyperboloidal foliations.

Outline

1. Model: equivariant wave maps on the $(d+1)$ – dimensional wormhole
2. Static solutions
3. Soliton resolution for $d \geq 3$ (B-Kahl, Rodriguez)
4. Soliton resolution for $d = 2$ (work in progress with Jendrej and Maliborski)

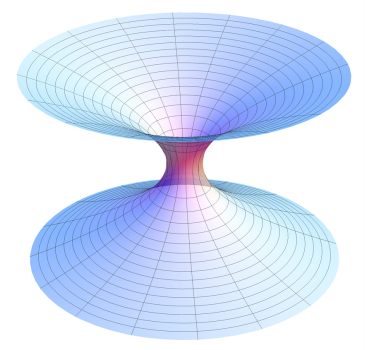
Related work: soliton resolution for Yang-Mills (B-Cownden-Maliborski)

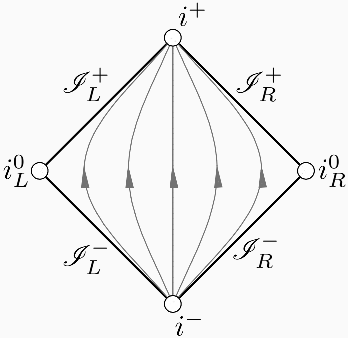
Wormhole

- Manifold $M = \{t \in \mathbb{R}, (r,\omega) \in \mathbb{R} \times \mathbb{S}^{d-1}\}$ with metric

$$ds^2 = -dt^2 + dr^2 + f^2(r)\, d\omega^2_{S^{d-1}}, \qquad f(r) = \sqrt{r^2 + a^2}$$

- Hypersurfaces $t =$ const have two asymptotically flat ends at $r \to \pm\infty$ connected by a neck of area $4\pi a^2$ at $r = 0$.





Wave maps on a wormhole

- A map $X : M \mapsto N$ from a Lorentzian manifold $(M, g_{\alpha\beta})$ into a Riemannian manifold (N, G_{AB}) is the wave map if it is a critical point of the action

$$S[X] = \int_M g^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B G_{AB}$$

- The wave map equation

$$\Box_g X^A + \Gamma^A_{BC}(X) \partial_\alpha X^B \partial_\beta X^C g^{\alpha\beta} = 0$$

  where Γ^A_{BC} are the Christoffel symbols of G_{AB}.

- Our model:
  - Domain M: the wormhole with metric

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)\, d\omega^2_{S^{d-1}}$$

  - Target: $N = \mathbb{S}^d$ with the round metric $ds^2 = dU^2 + \sin^2 U\, d\theta^2_{S^{d-1}}$

- Equivariant ansatz: we assume that $U = U(t, r), \theta = \chi_k(\omega)$, where $\chi_k : S^{d-1} \mapsto S^{d-1}$ is a harmonic map with eigenvalue $\lambda_k = k(k + d - 2)$

- Equivariant wave map equation

$$U_{tt} = U_{rr} + \frac{(d-1)r}{r^2+a^2} U_r - \frac{\lambda_k}{2} \frac{\sin(2U)}{r^2+a^2}$$

- The length scale a removes the singularity at $r=0$, ensuring global-in-time regularity. Below we set $a=1$.
- Let $r = \sinh x$ and $u(t,x) = U(t,r)$. Then

$$\cosh^2 x\, u_{tt} = u_{xx} + (d-2) \tanh x\, u_x - \frac{\lambda_k}{2} \sin(2u)$$

- Conserved energy

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\cosh^2 x\, u_t^2 + u_x^2 + \lambda_k \sin^2 u\right) (\cosh x)^{d-2}\, dx$$

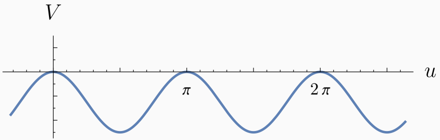
- Finite energy requires that $u(t,-\infty) = m\pi$, $u(t,\infty) = n\pi$ $(m,n \in \mathbb{Z})$. We choose $m=0$ so n determines the topological sector (degree).
- Our aim is describe the asymptotic behavior of solutions for $t \to \infty$.

Static solutions

- Static solutions $u(x)$ satisfy the ODE

$$u'' + (d-2)\tanh x\, u' - \frac{\lambda_k}{2}\sin(2u) = 0$$

- For $d \geq 3$ this describes a particle in the potential $V = -\frac{\lambda_k}{2}\sin^2 u$ subject to 'time'-dependent friction.



- By elementary shooting argument, for each n there exists a unique solution $u = Q_n(x)$ such that $Q_n(-\infty) = 0$ and $Q_n(\infty) = n\pi$.
- Q_n is a minimizer of energy for given $n \Rightarrow$ Lyapunov stability

Static solutions in $d = 2$

$$u'' - \frac{k^2}{2}\sin(2u) = 0$$

- In the degree one sector we have

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u'^2 + k^2\sin^2 u\right)\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(u' - k\sin u\right)^2\,dx + 2k \geq 2k$$

- This inequality is saturated on the kink solution $u = Q(x) = 2\arctan\left(e^{kx}\right)$
- $Q(x) \xrightarrow{translation} Q_c(x) = Q(x - c)$
- Kink is a degenerate minimizer of energy $\Rightarrow$ orbital stability
- For $n > 1$ there are no static solutions and $E(u) > n\,E(Q) = 2nk$.

Hyperboloidal formulation (due to Friedrich and Zenginoğlu)

- Let $t = s + \cosh x$. Then

$$u_{ss} + 2\sinh x\, u_{sx} + (\cosh x + (d-2)\tanh x \sinh x)\, u_s = u_{xx} + (d-2)\tanh x\, u_x - \frac{\lambda_k}{2}\sin(2u)$$

- Asymptotic behaviour for smooth initial data of degree n

$$u(s,x) \sim \begin{cases} b_-(s) e^{\frac{d-1}{2}x} & \text{for} \quad x \to -\infty, \\ n\pi + b_+(s) e^{-\frac{d-1}{2}x} & \text{for} \quad x \to \infty \end{cases}$$

- Multiplying by $(\cosh x)^{d-2} u_s$, one gets $\partial_s \rho + \partial_x f = 0$, where

$$\rho = \left[u_s^2 + u_x^2 + \lambda_k \sin^2 u\right] (\cosh x)^{d-2}, \quad f = \left[\sinh x\, u_s^2 - u_s u_x\right] (\cosh x)^{d-2}$$

- Defining the energy $\mathscr{E}(u) = \int_{-\infty}^{\infty} \rho \, dx$, we get the energy loss formula

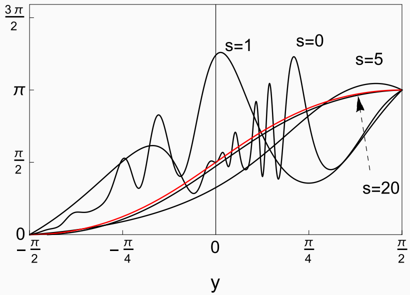
$$\frac{d\mathscr{E}}{ds} = -\dot{b}_-^2 - \dot{b}_+^2$$

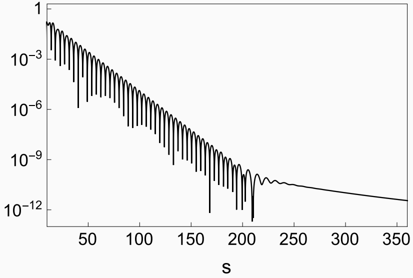
- What is the limit $\mathscr{E}_\infty = \lim_{s\to\infty} \mathscr{E}(u(s))$?

Soliton resolution ($d = 3$)

For any smooth initial data of degree n there exists a unique smooth global solution which asymptotically converges to the kink Q_n.

- Heuristics and numerical evidence (B-Kahl 2015)
- Proof (Rodriguez 2016)





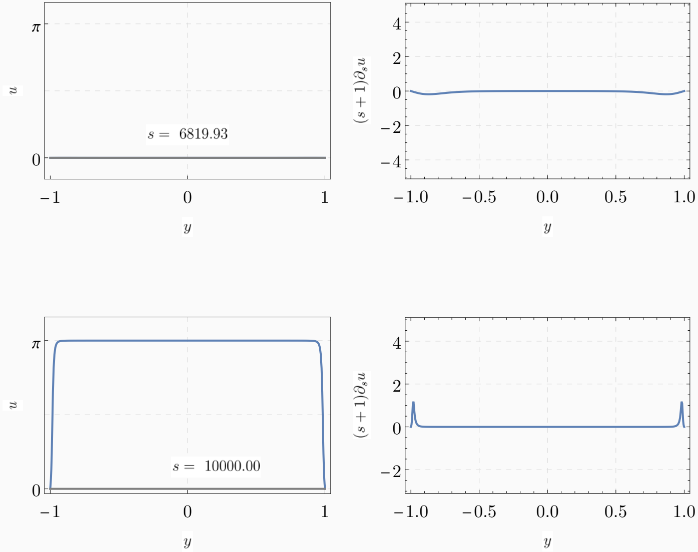
Soliton resolution conjecture in $d = 2$

- Joint work with Jacek Jendrej and Maciej Maliborski (in preparation)
- We conjecture that solutions converge either to zero (if $n = 0$) or to a N-chain of kinks and antikinks

$$u(s,x) \simeq \sum_{j=1}^{N} \sigma_j Q(x - c_j(s)), \qquad |c_{j+1}(s) - c_j(s)| \to \infty$$

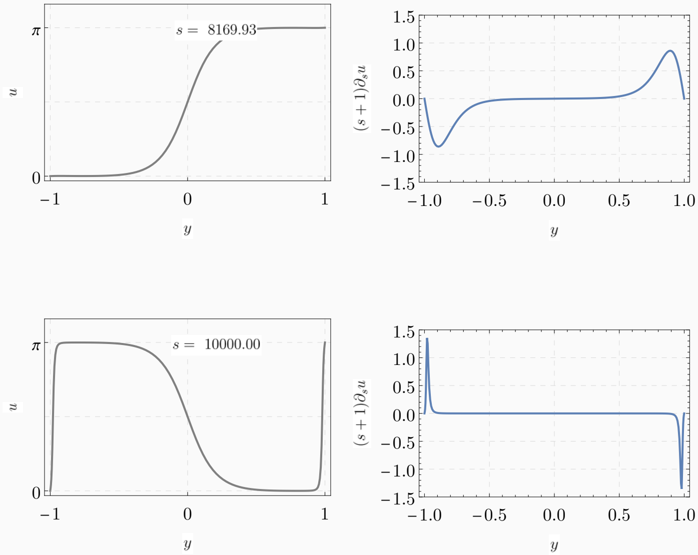
  Here the signs $\sigma_j = \pm 1$ correspond to kinks and antikinks, respectively.
- $\mathcal{E}_\infty = N E(Q) = 2Nk$. Here $N = n + 2m$, where m is the number of kink-antikink pairs created in the evolution.
- Analogous result for equivariant wave maps from $2+1$ Minkowski spacetime into the two-sphere was proved by Jendrej and Lawrie 2021.
- Below we consider the $n = 0$ and $n = 1$ cases.

$n = 0$



π
u
s = 6819.93
0
−1
0
1
y
4
2
0
−2
−4
(s + 1)∂ₛu
−1.0
−0.5
0.0
0.5
1.0
y
π
u
s = 10000.00
0
−1
0
1
y
4
2
0
−2
(s + 1)∂ₛu
−1.0
−0.5
0.0
0.5
1.0
y

$n = 1$



s = 8169.93
π
0
−1
0
1
u
y
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
(s + 1)∂_s u
y
s = 10000.00
π
0
−1
0
1
u
y
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
(s + 1)∂_s u
y

Asymptotically static solutions

- As the first step in understanding the global phase portrait, we determine the behavior of solutions at the threshold of the kink-antikink creation. They separate basins of attractions of different N-chains.
- The threshold solutions are asymptotically static when viewed in the coordinates (t,x) i.e. their kinetic energy goes to zero for $t \to \infty$.
- Finite dimensional analogue: Newton's equation

$$\ddot{x}(t) = F(x(t)) \quad \text{with} \quad \dot{x}(t)^2 \to 0 \quad \text{as} \quad t \to \infty$$

  Here asymptotically static solutions describe parabolic motions.
- The asymptotically static kink-antikink pair (called the two-bubble) was proved to exist for equivariant wave maps from $2+1$ Minkowski spacetime into the two-sphere by Jendrej and Lawrie 2020.

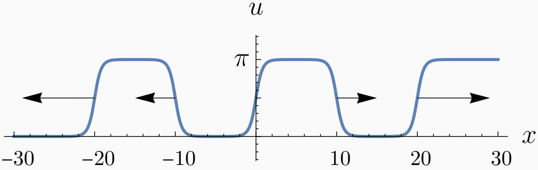
Conjecture

There exist asymptotically static N-chains the form

$$u(t,x) = \begin{cases} Q(x) + \sum_{j=1}^{K} (-1)^j \left[Q(x+c_j(t)) + Q(x-c_j(t))\right], & N = 2K+1 \\ \sum_{j=1}^{K} (-1)^j \left[Q(x+c_j(t)) - Q(x-c_j(t))\right], & N = 2K \end{cases}$$

where $c_j(t) \to \infty$ and $c_1 \ll c_2 \ll \cdots \ll c_N$.

Example $N = 5$



- We have numerical and analytic evidence for $N = 2, 3, 4, 5$.

Method of collective coordinates

- Plugging the $(2K+1)$-chain into the Lagrangian (for $k=2$)

$$\mathscr{L} = \frac{1}{2} \int_{-\infty}^{\infty} \left(\cosh^2 x \, u_t^2 - u_x^2 - 4 \sin^2 u \right) dx$$

  we get the reduced Lagrangian (where $c_0 = 0$)

$$L \approx \sum_{j=1}^{K} e^{2c_j} \dot{c}_j^2 - \sum_{j=1}^{K} e^{-2(c_j - c_{j-1})}$$

- This approach omits radiation, hence it is expected to work only if radiation is negligible.

- Let $r_j = e^{c_j}$. Then we get the K-body problem on a half-line

$$L \approx \sum_{j=1}^{K} \dot{r}_j^2 - V, \qquad V = -\sum_{j=1}^{K} \frac{r_{j-1}^2}{r_j^2}$$

- Equations of motion (where $r_0 = 1$ and $r_{K+1} = \infty$)

$$\ddot{r}_j = -\frac{r_{j-1}^2}{r_j^3} + \frac{r_j}{r_{j+1}^2}, \qquad j = 1, \ldots, K$$

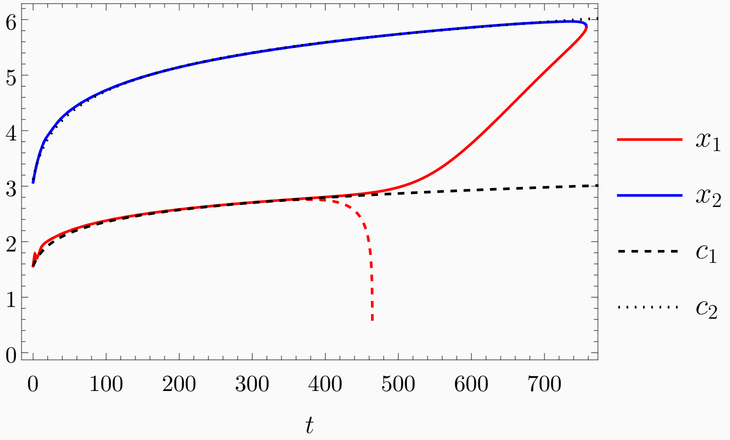
- For each K there exists a zero energy asymptotically self-similar solution such that for $t \to \infty$

$$r_j(t) \sim (At)^{\frac{j}{K+1}}, \quad \text{where} \quad A = \frac{K+1}{\sqrt{K}}$$

  - For $K = 1$ this solution is exact.
  - For $K \geq 2$ the solution can be constructed by a perturbation method.

- Threshold solutions of the PDE (for $N = 2, 3, 4, 5$) are very well approximated for $t \to \infty$ by the reduced ODE model.

$N = 5$



Summary

- The soliton resolution for $d+1$ equivariant maps into S^d is well understood in $d \geq 3$.
- For $d = 2$ there are only partial results. Quantitative agreement between the PDE numerics and the ODE approximation is a promising first step.
- The hyperboloidal approach has been helpful in numerical computations of long time dynamics, however has not been used as an analytic tool.
- Many open mathematical problems, e.g. modulation analysis of the interaction of the kink with radiation

$$u(t,x) = Q_{c(t)}(x) + \eta(t,x)$$