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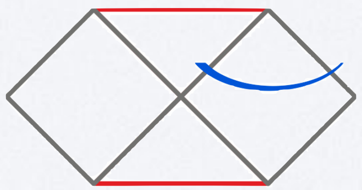
Danmarks Grundforskningsfond Danish National Research Foundation

HYPERBOLOIDAL APPROACH ON BH SPACETIMES: THE MINIMAL GAUGE

ArXiv: 2307.15735 - Royal Society Phil. Trans. A
"At the interface of asymptotic conformal methods and analysis in GR"



Rodrigo Panosso Macedo

OUTLINE

- Motivation
- The development of the minimal gauge
- Practical “recipe”: change signs in the tortoise coordinate
- Conclusion
- Projects

MOTIVATION

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(Umberto Eco - Foucault's Pendulum)

MOTIVATION

"That's why the Pendulum disturbs me. It promises the infinity, but where to put the infinity is left to me. So it isn't enough to worship the Pendulum; you still have to make a decision, you have to find the best point for it. And yet…"

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MOTIVATION

"That's why the Conformal Factor disturbs me. It promises the infinity, but where to put the infinity is left to *us*. So it isn't enough to worship the Conformal Factor; *we* still have to make a decision, *we* have to find the best point for it. And yet…"

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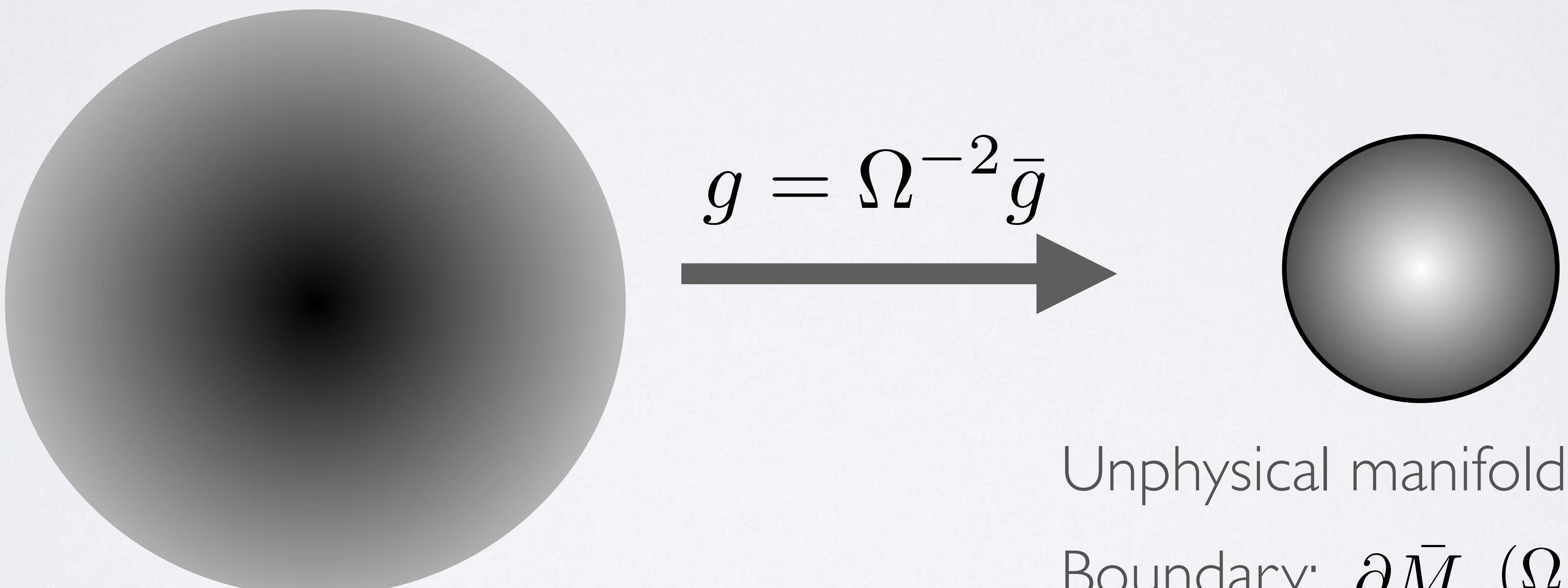

Unphysical manifold $(\bar{\mathcal{M}}, \bar{g})$

Boundary: $\partial\bar{M}$ $(\Omega = 0)$

Physical manifold $(\mathcal{M}, g)$

Decision: coordinates for hyperboloidal slices in black-hole spacetimes

BLACK HOLE GAUGES

- Schwarzschild (Boyer-Lindquist) coordinates: (t, r, θ, φ)
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Constraint Equations + Non-Linear Evolutions

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$$ds^2 = -a(r)dt^2 + \frac{dr^2}{b(r)} + r^2 d\omega^2$$

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$$ds^2 = \Omega^{-2} d\bar{s}^2$$

$$\Omega = \frac{\sigma}{\lambda}$$

$$d\bar{s}^2 = \Xi(\sigma)\left[-p(\sigma)d\tau^2 + 2\gamma(\sigma)d\tau d\sigma + w(\sigma)d\sigma^2\right] + \rho(\sigma)^2 d\omega^2$$

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$$\Omega = \frac{\sigma}{\lambda}$$

$d\bar{s}^2 = \Xi$ ... $d\omega^2$

Decision #0

Use conformal factor as radial coordinate: $\sigma = \lambda\Omega$

HYPERBOLOIDAL METRIC

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<u>Radial degrees of freedom</u>

$\rho(\sigma)$: Conformal areal radius $\qquad \beta(\sigma) := \rho(\sigma) - \sigma\rho'(\sigma)$

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Spacetime properties

$\mathcal{F}(\sigma) = \sqrt{a(r)b(r)}$

$\zeta(\sigma) = \sqrt{a(r)/b(r)}$

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Horizons: $\mathcal{F}(\sigma_h) = 0$

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Fancy Metric: $\zeta(\sigma) \neq 1$

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HYPERBOLOIDAL GEOMETRY

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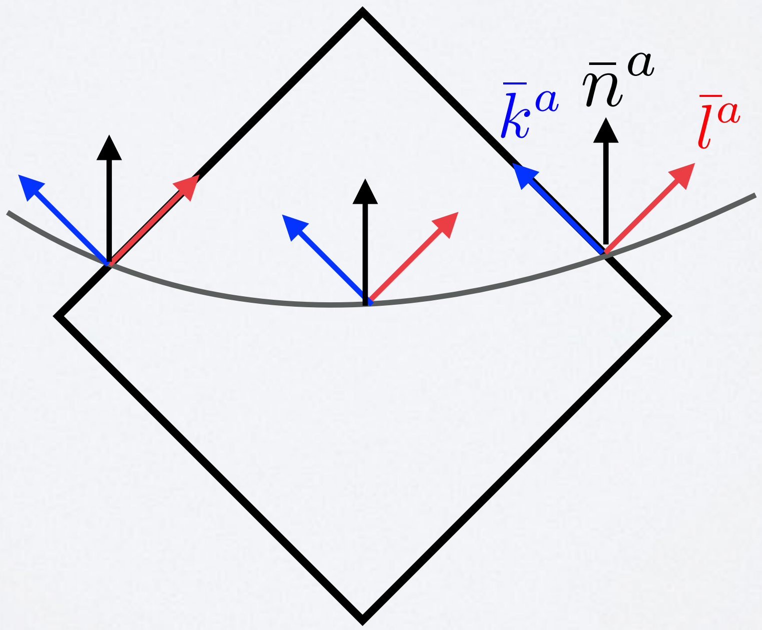

SCHWARZSCHILD

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\omega^2, \quad f(r) = 1 - \frac{r_h}{r}$$

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$$\phi(r) = \underbrace{r^{2i\omega}\,(r-1)^{-i\omega}\,e^{2i\omega(r-1)}}_{\text{BC}}\;\underbrace{\bar{\phi}_{\mathrm{L}}(r)}_{\text{regular}}$$

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$$(t, r, \theta, \varphi)$$

$$-\psi_{,tt} + \psi_{,r_* r_*} - V(r)\psi = 0$$

Fourier Transf. $\quad \psi(t,r) = e^{-i\omega t}\phi(r)$

$$\phi_{,r_* r_*} - \left(V(r) - \omega^2\right)\phi = 0$$

Ansatz ODE theory

$$\phi(r) = \underbrace{r^{2i\omega}\,(r-1)^{-i\omega}\,e^{2i\omega(r-1)}}_{\mathrm{BC}}\;\underbrace{\bar{\phi}_{\mathrm{L}}(r)}_{\mathrm{regular}}$$

Taylor Expansion (Horizon)

$$\bar{\phi}_{\mathrm{L}}(r) = \sum_{n=0}^{\infty} a_n \left(1 - \frac{1}{r}\right)^n$$

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Taylor Expansion (Horizon)

$$\bar{\phi}(\sigma) = \sum_{n=0}^{\infty} \bar{a}_n (1-\sigma)^n$$

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Take Away Message (M. Ansorg, RPM 2016)

Leaver's regular field is the frequency-domain representation of the hyperboloidal perturbation field $\bar{\psi}(\tau, \sigma)$ in the minimal gauge

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Decision #1

Follow Ansorg's Recipe

1. Ingoing null coordinates
2. Integrate asymptotic behaviour of outgoing light rays
3. Choose integration constant as time coordinate
4. Simple compactification

REISSNER-NORDSTRÖM

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\omega^2, \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

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2. Consider asymptotic behaviour of outgoing light rays

$$\frac{dv}{dr} = \frac{2}{f(r)} \approx 2\left(1 + \frac{2M}{r}\right)$$

$$v = 2r + 4M \ln\left(\frac{r}{r_h}\right) + C$$

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"you still have to make a decision, you have to find the best point for it. And yet…"

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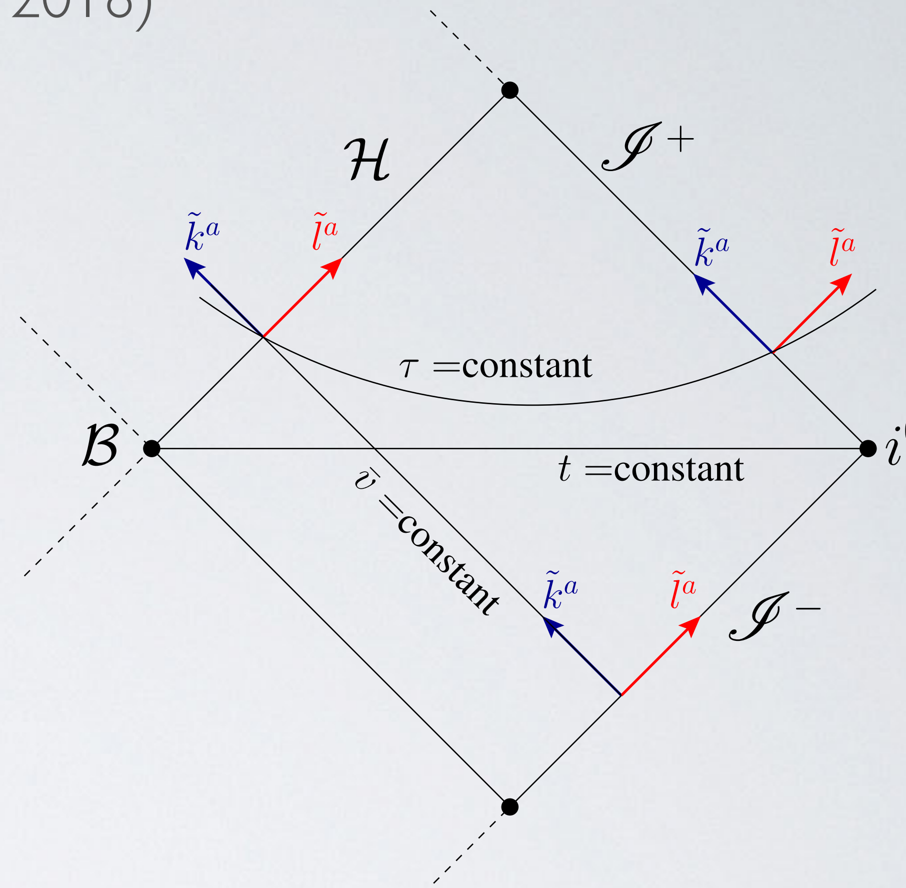
LEAVER EXPRESSION'S SEEMS SIMPLER

BUT LEAVER'S APPROACH FAILS IN A DIRECT LIMIT TO THE EXTREMAL CASE, WHEREAS THIS RECIPE DOESN'T

MINIMAL GAUGE

(RPM, J.L. Jaramillo, M. Ansorg 2018)



4. Simple compactification

MINIMAL GAUGE

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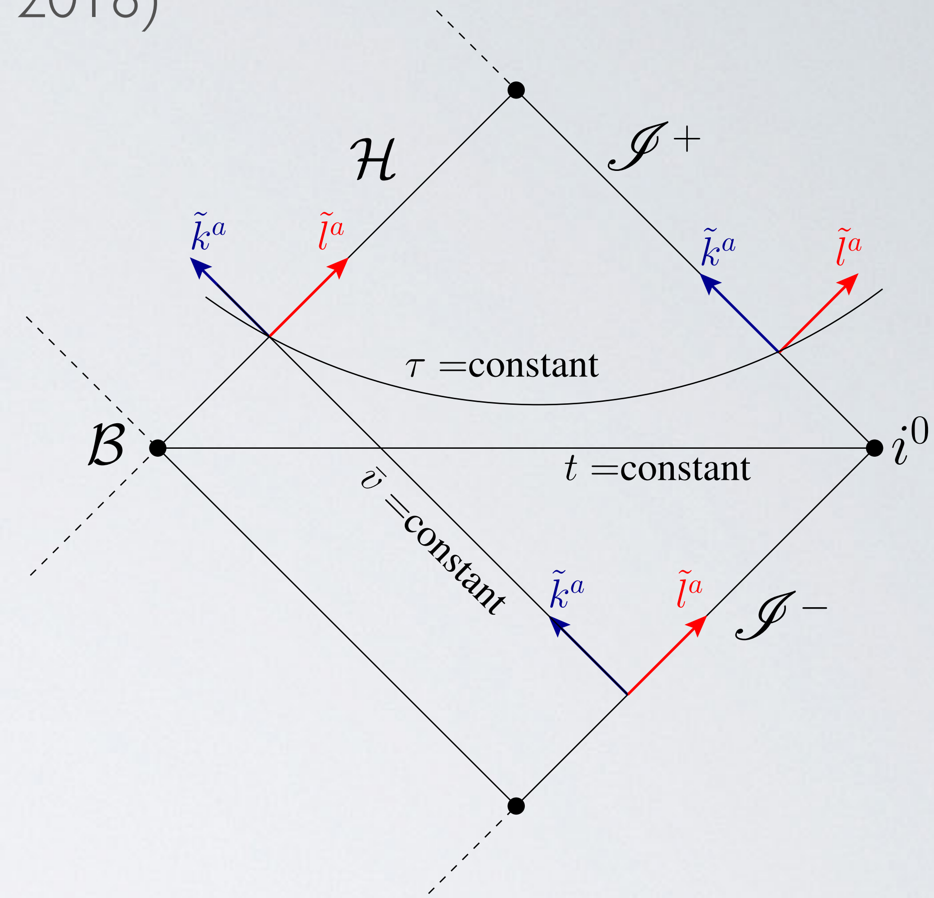

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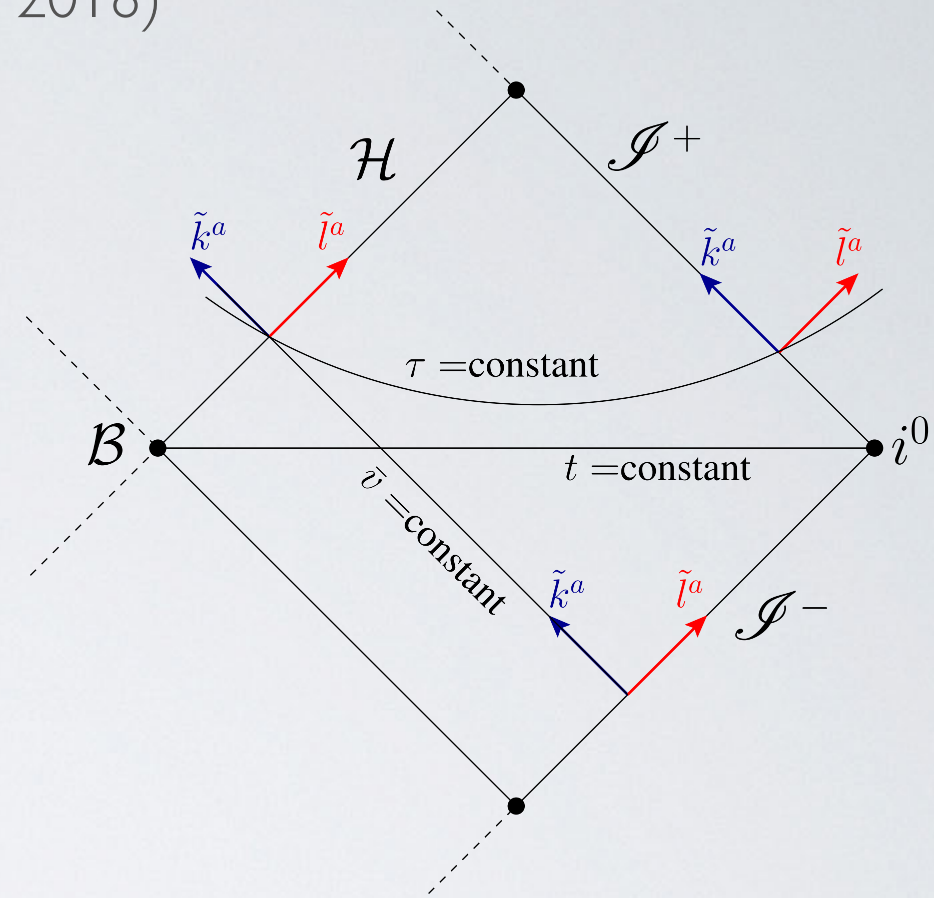
2. Compactified hyperboloidal Ansatz



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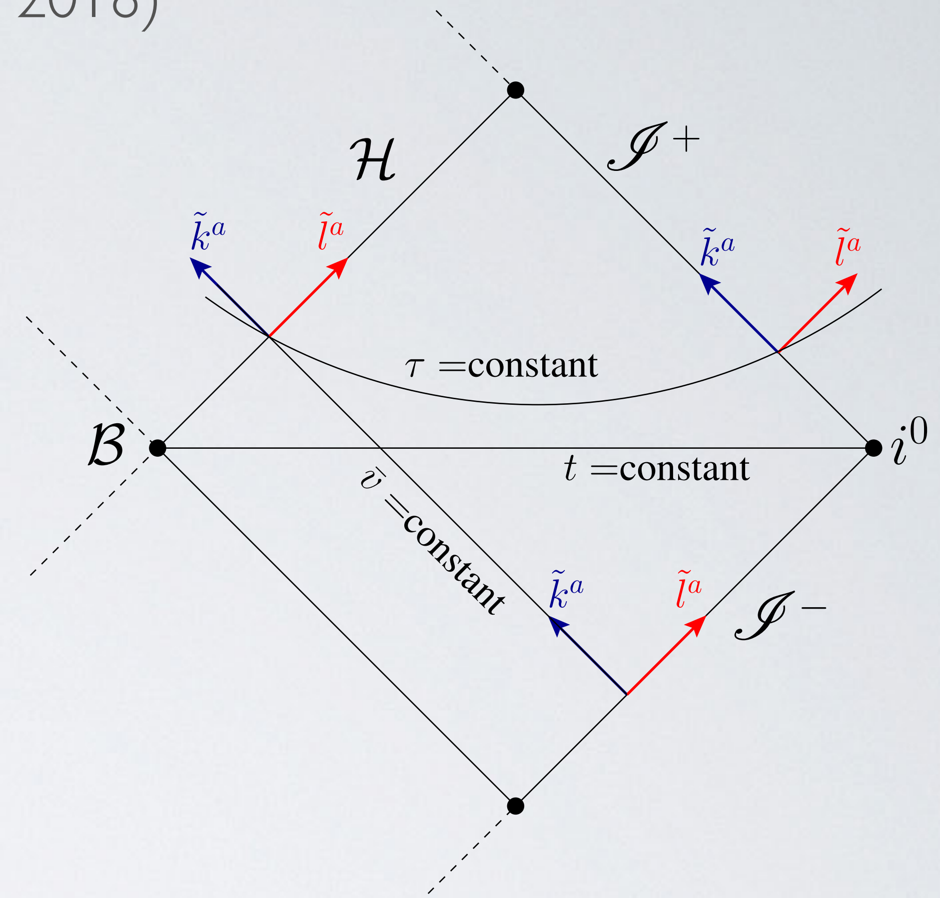

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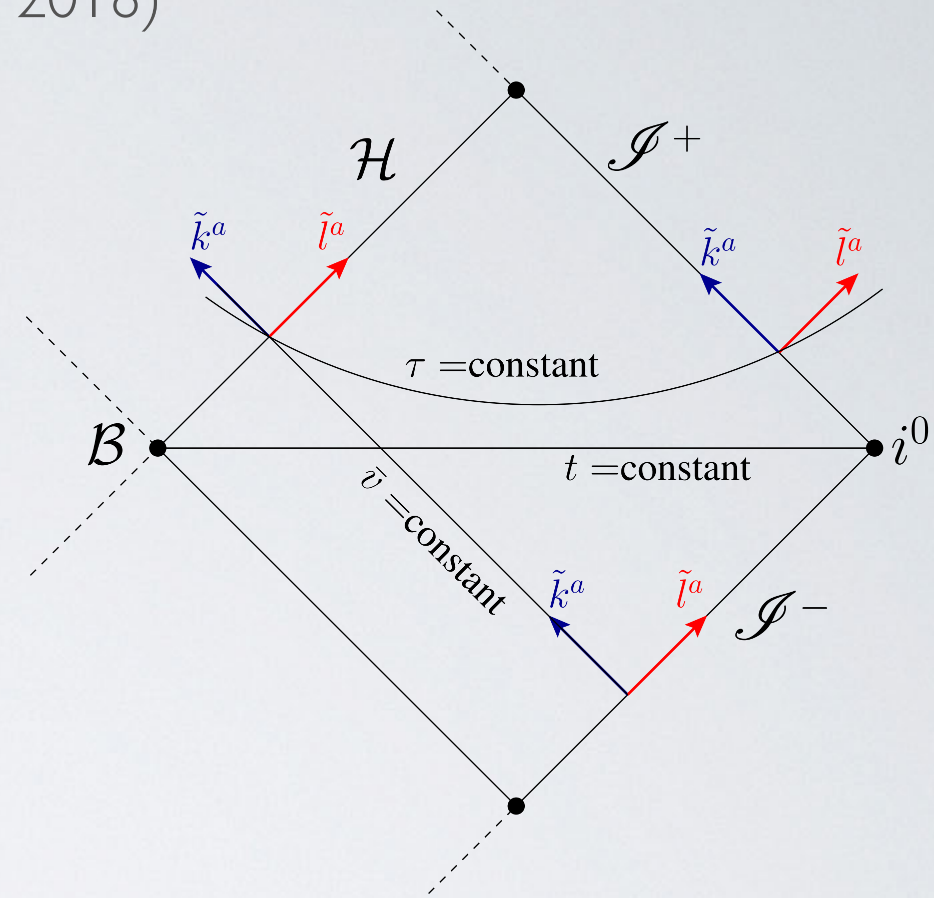

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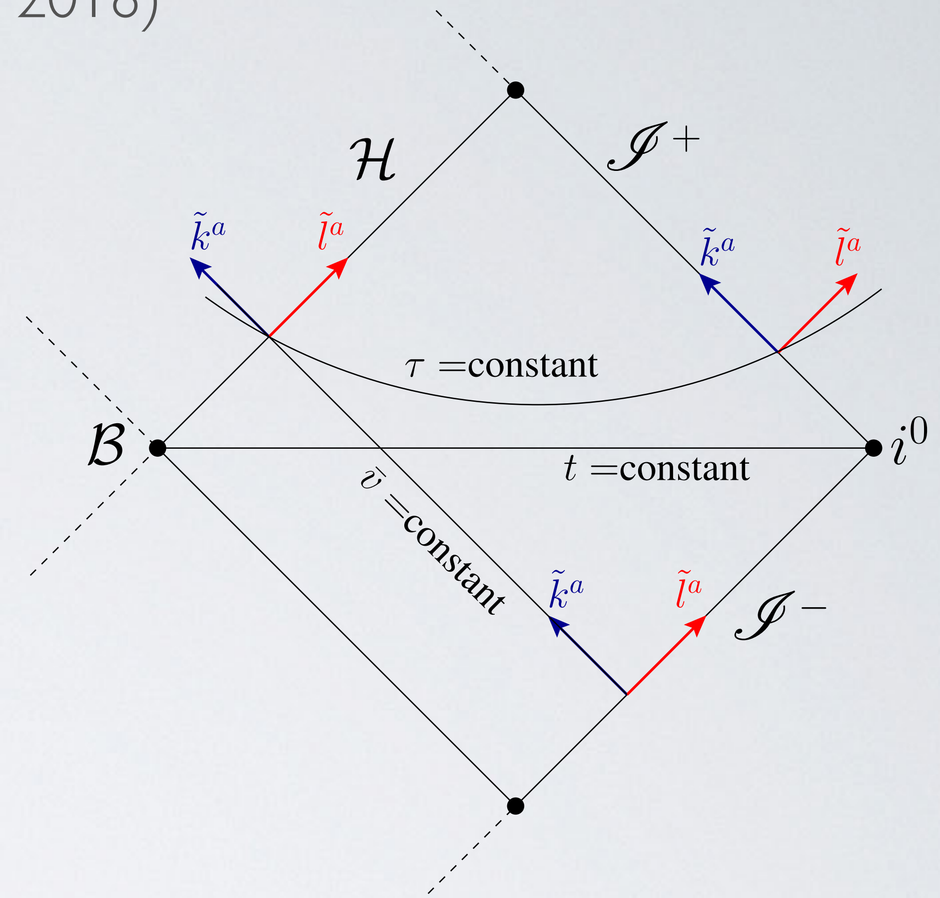
$$\Updownarrow$$

align ∂_τ with $\bar{k}^a$ and ensure $\bar{l}^a$ is regular



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$$v = \lambda\Big(\tau - h(\sigma)\Big) \qquad r = \lambda\frac{\rho(\sigma)}{\sigma} \qquad \Big(\beta(\sigma) := \rho(\sigma) - \sigma\rho'(\sigma)\Big)$$

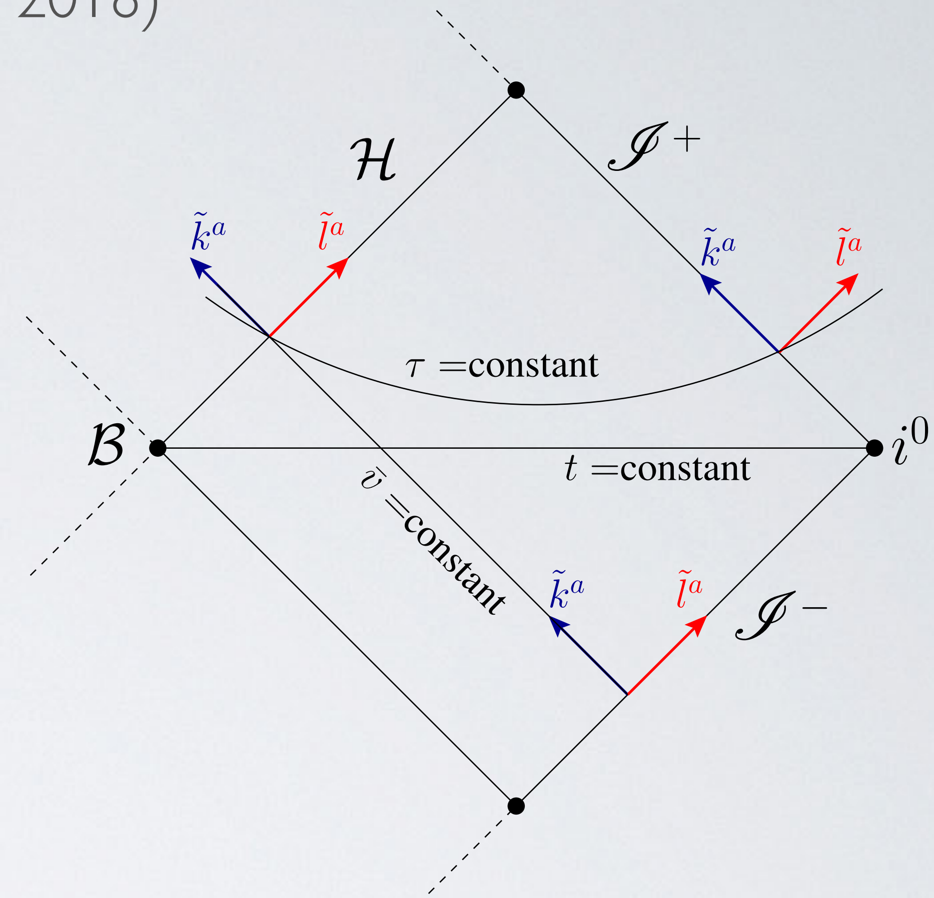
3. Asymptotic behaviour of outgoing light rays

$$\Updownarrow$$

align ∂_τ with $\bar{k}^a$ and ensure $\bar{l}^a$ is regular

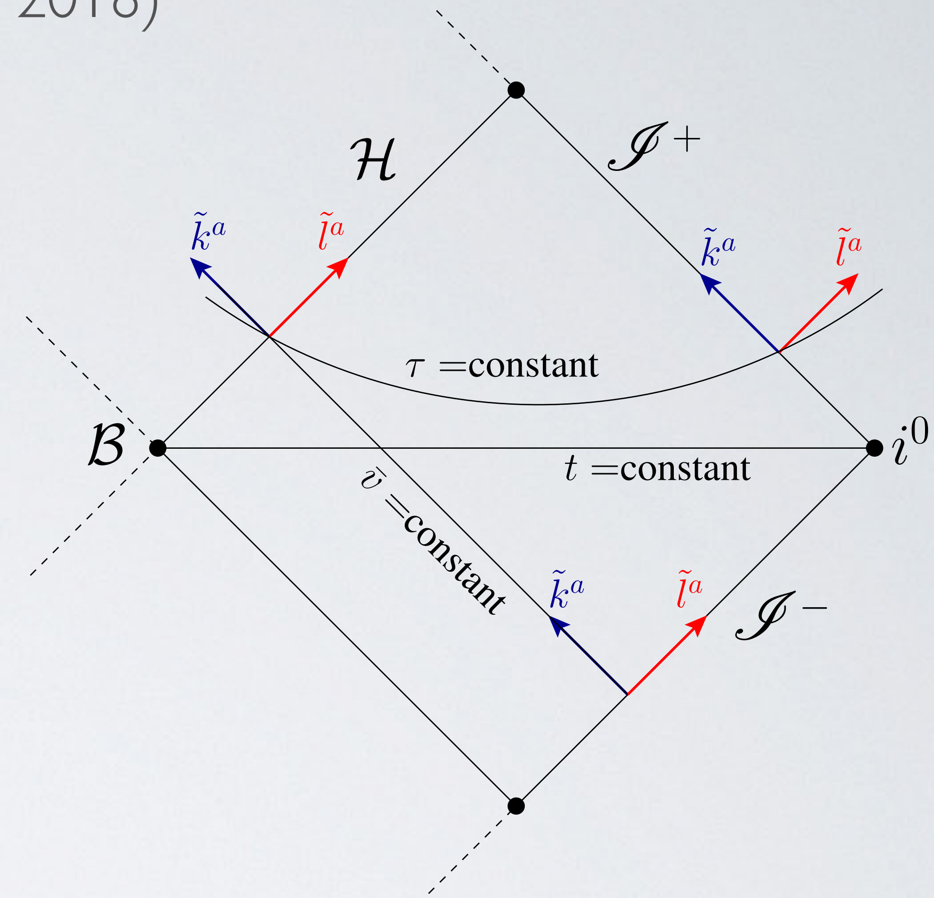
$$h(\sigma) = \underbrace{h_0(\sigma)}_{\text{minimal requirements}} + \underbrace{A(\sigma)}_{\text{regular}}$$

4. Simple compactification



MINIMAL GAUGE

(RPM, J.L. Jaramillo, M. Ansorg 2018)



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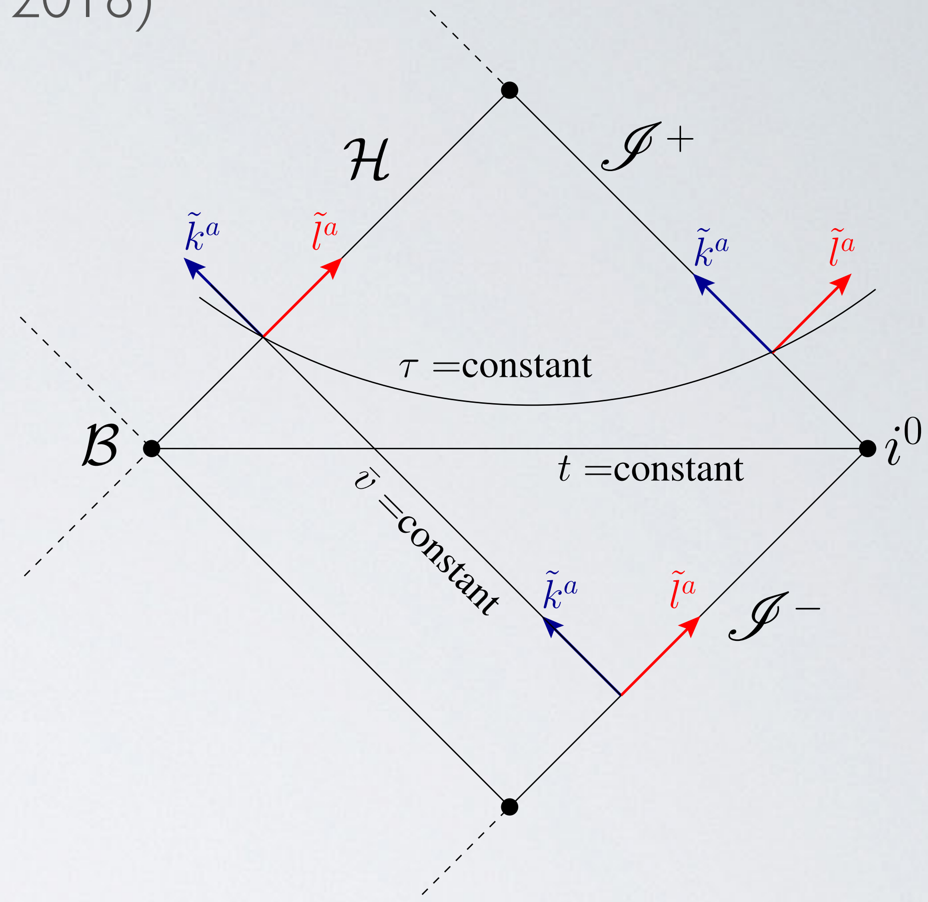

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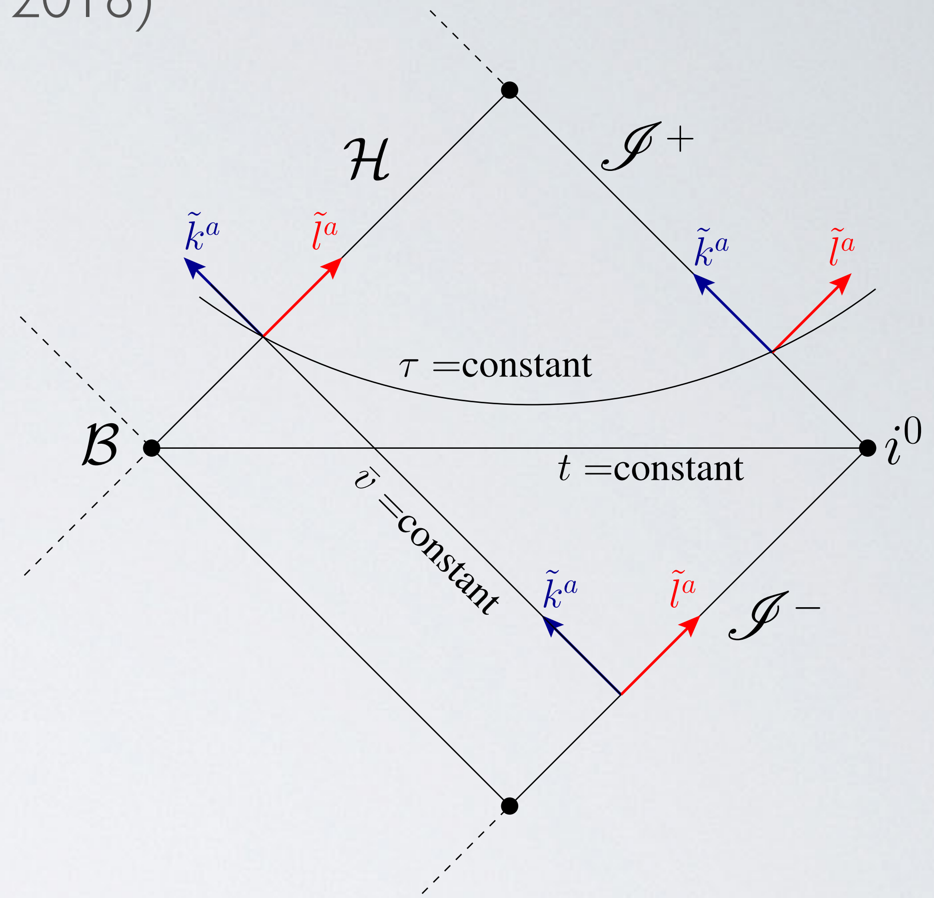

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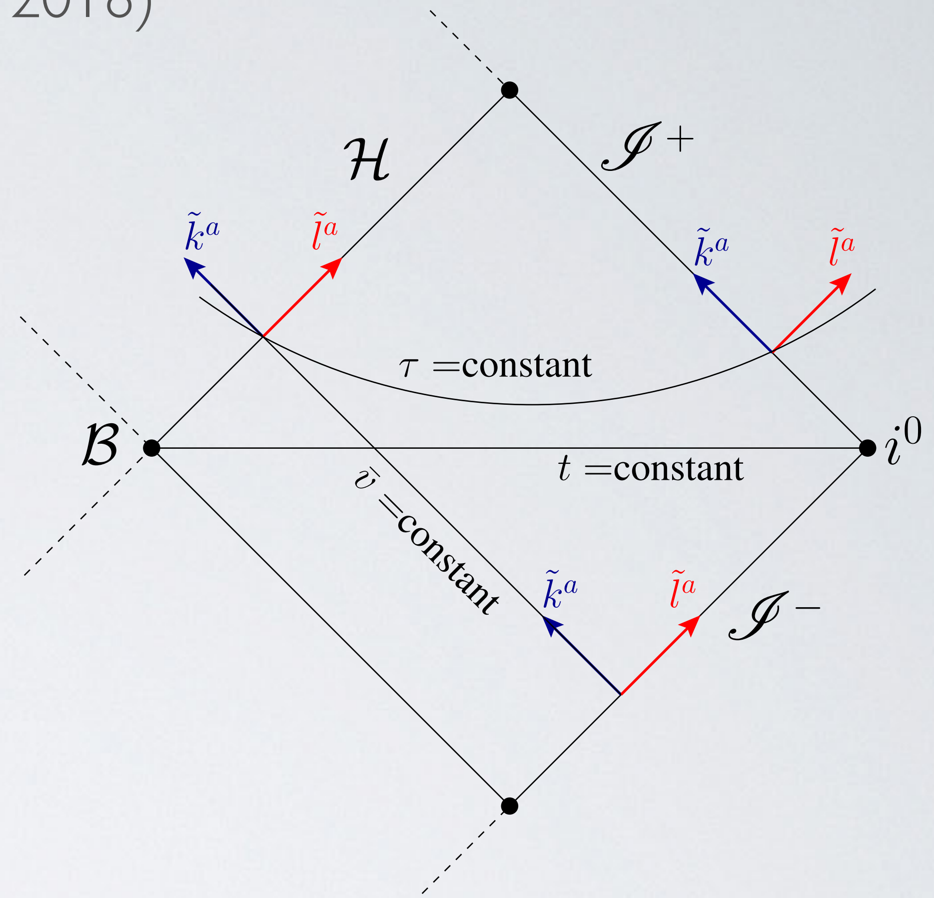

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REISSNER-NORDSTRÖM

(And Kerr - RPM 2019)

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(A) radial fix

$$\rho_1 = 0$$

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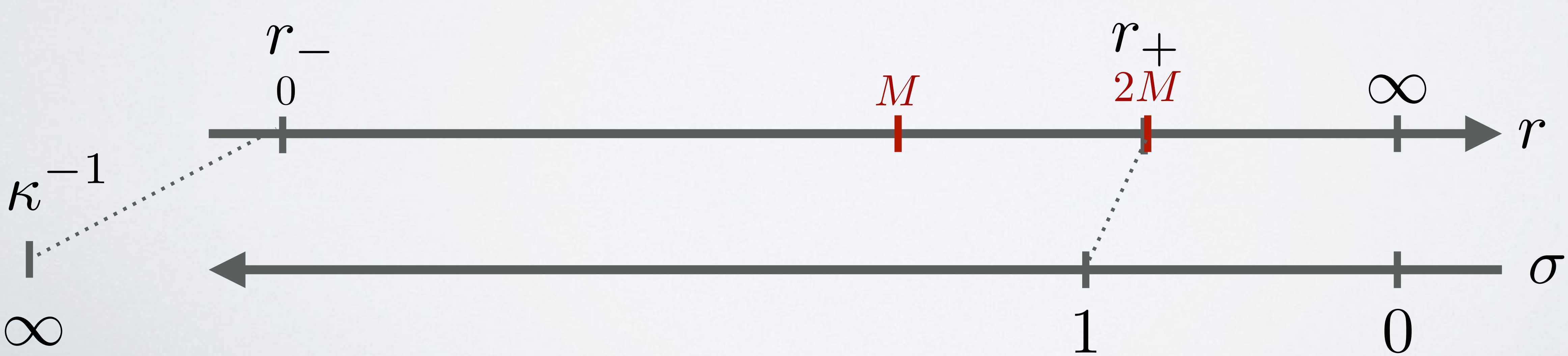
(charge/spin parameter)



Schwarzschild $\kappa = 0$

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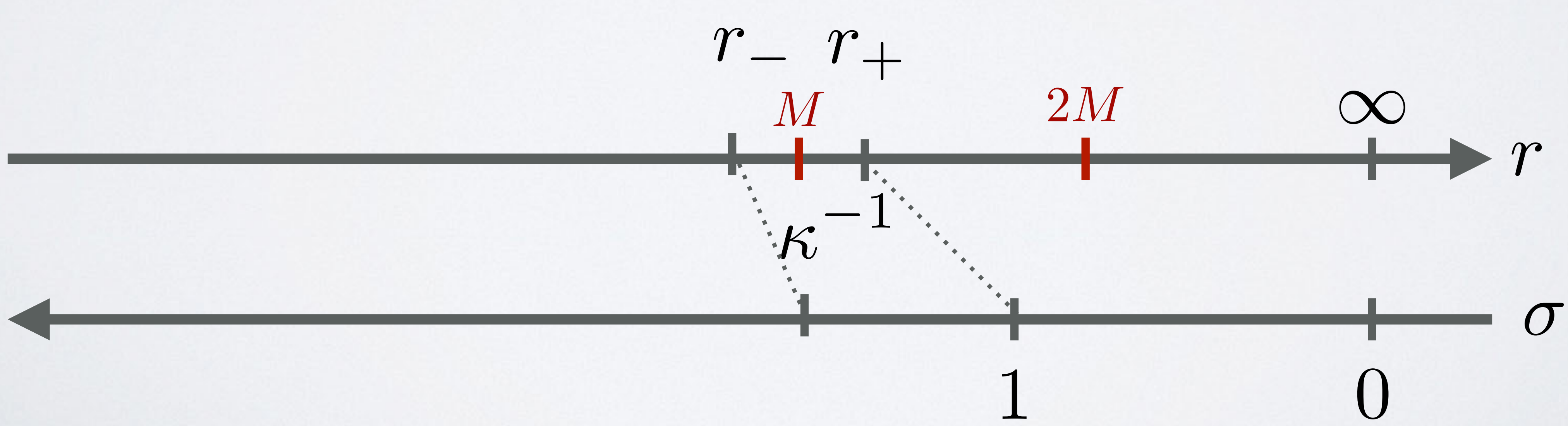

"Charged" $0 < \kappa < 1$

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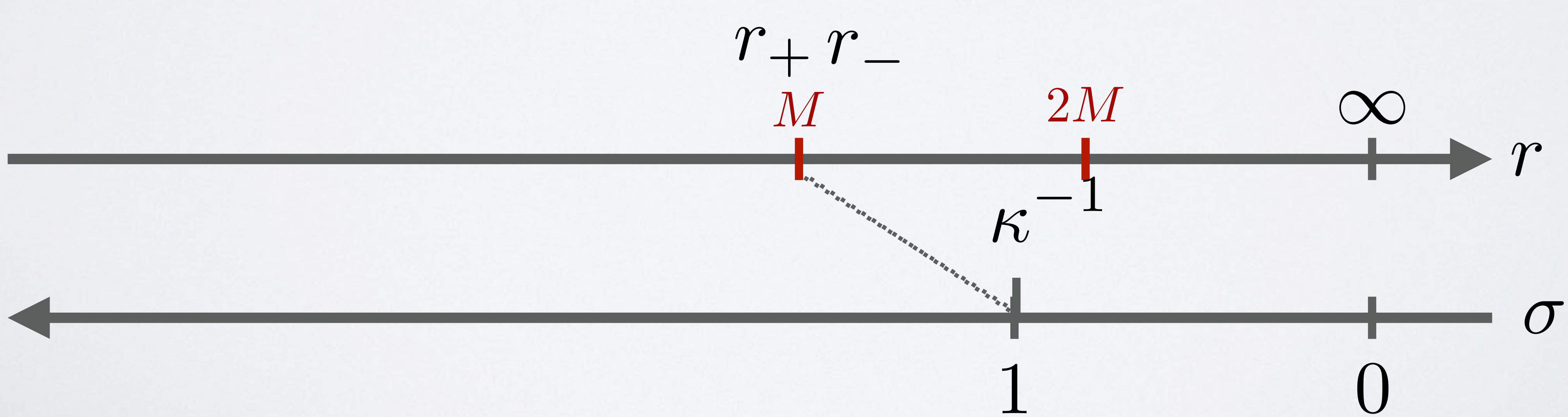

"Extremal" $\kappa = 1$

REISSNER-NORDSTRÖM

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(charge/spin parameter)

(B) Cauchy horizon fix

$$\rho_1 = \kappa$$

$$r(\sigma) = \frac{r_h}{\sigma}\Big(1 - \kappa(1-\sigma)\Big)$$

REISSNER-NORDSTRÖM

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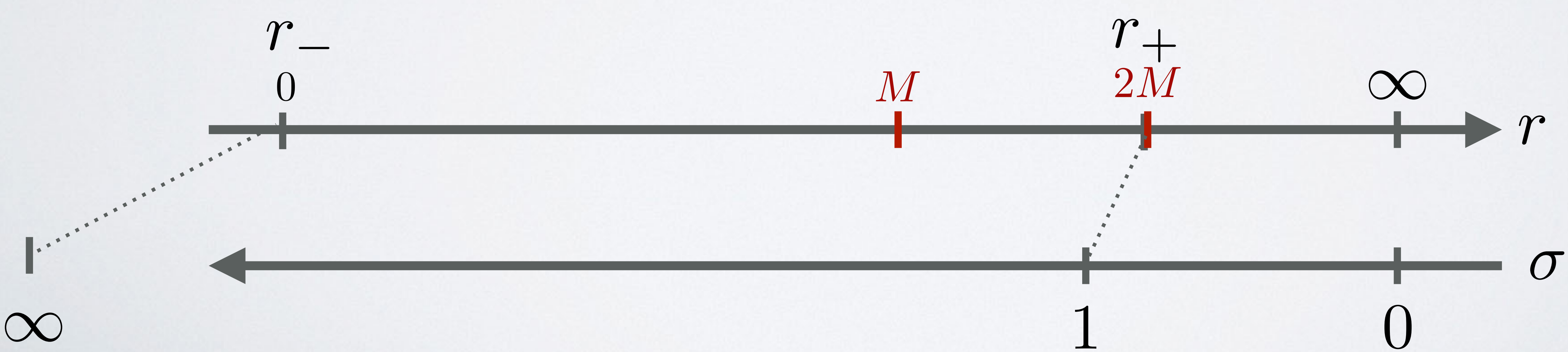

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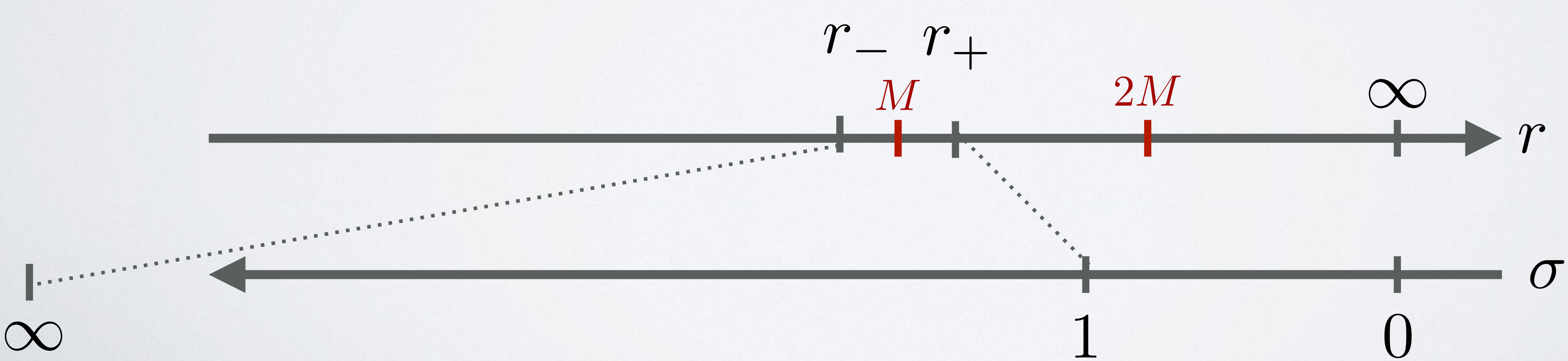

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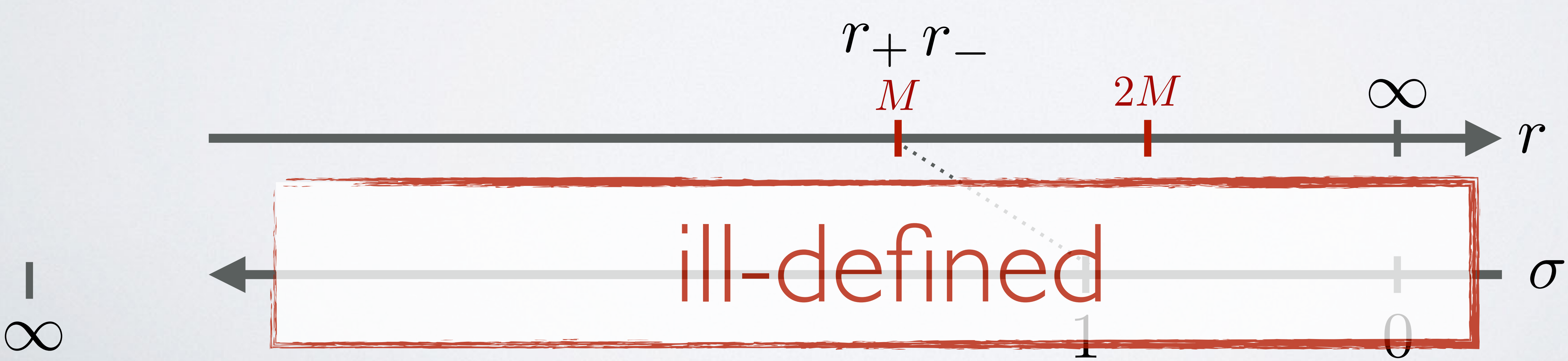

“Extremal” $\kappa = 1$

REISSNER-NORDSTRÖM

(B) Cauchy horizon fixing: the extremal limit

$$r = \frac{r_h}{\sigma}\left(1 - \kappa(1-\sigma)\right) \xrightarrow{\kappa=1} r_h$$

$$d\tilde{s}^2 = (\cdots)(1-\kappa)^2 d\tau^2 + (\cdots)(1-\kappa)d\tau d\sigma + (\cdots)(1-\kappa)d\sigma^2 + \rho^2 d\omega^2$$

REISSNER-NORDSTRÖM

(B) Cauchy horizon fixing: the extremal limit

$$r = \frac{r_h}{\sigma}\Big(1 - \kappa(1-\sigma)\Big) \xrightarrow{\kappa=1} r_h \qquad T = (1-\kappa)\tau \xrightarrow{\kappa=1} 0$$

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REISSNER-NORDSTRÖM

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$$\Big\downarrow \quad \kappa = 1$$

$$d\tilde{s}^2_{\text{ext}} = -(1-\sigma)dT^2 + dT d\sigma + \sigma^2 d\omega^2$$

The Robinson-Bertotti metric (near-horizon geometry): $AdS_2 \times S^2$

B. Bertotti, Phys. Rev 116 1331 (1959)

R.I., Bull Acad. Polon. Sci. 7 351 (1959)

R.P. Geroch, Commun. Math. Phys. (1969)

F.M.Paiva, M.J. Rebouças, M.A.H. MacCalllum Quant. Grav. 10 1165 (1993)

S.M. Carrol, M.C. Johnson, L.Randall, JHEP 109 (2009)

I. Bengtsson, S. Holst, E. Jakobsson CQG 31 205008 (2014)

QUASINORMAL MODES

(Leaver 1990) $\lambda = r_h \Leftrightarrow 2M = 1$

$$\phi(r) = \underbrace{e^{sH(\sigma)}}_{\text{BC}} \; \underbrace{\bar{\phi}(\sigma)}_{\text{regular}}$$

Radial transformation in the (radial fix) Minimal Gauge

Height Function in the Minimal Gauge

$$\phi(r) = \underbrace{Z_\infty(r) Z_{r_h}(r) Z_{r_c}(r)}_{\text{BC}} \; \underbrace{\bar{\phi}_{\text{L}}(r)}_{\text{regular}}$$

THEY DON'T AGREE

LEAVER EXPRESSION'S SEEMS SIMPLER

BUT LEAVER'S APPROACH FAILS IN A DIRECT LIMIT TO THE EXTREMAL CASE, WHEREAS THIS RECIPE DOESN'T

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Leaver's regular field is the frequency-domain representation of hyperboloidal field in minimal gauge

Leaver's approach fails in a direct limit to the extremal case, because spacetime representation corresponds to discontinuous transition into the near-horizon geometry

BLACK HOLE + MATTER HALO

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{b(r)} + r^2 d\omega^2$$

$$a(r) = \left(1 - \frac{r_h}{r}\right) e^{\Upsilon(r)}, \quad b(r) = \left(1 - \frac{r_{\rm h}}{r}\right) B(r)$$

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- <u>Physical Interpretation:</u> Gravity coupled to anisotropic fluid. Geometry of supermassive black hole with a dark matter halo

(V. Cardoso, K. Destounis, RPM, F Duque, A Maselli, 2022)

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"you still have to make a decision, you have to find the best point for it. And yet…"

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$$d\bar{s}^2 = \Xi(\sigma)\left[-p(\sigma)d\tau^2 + 2\gamma(\sigma)d\tau d\sigma + w(\sigma)d\sigma^2\right] + \frac{r_h^2}{\lambda^2}d\omega^2$$

BLACK HOLE + MATTER HALO

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- Are hyperboloidal hypersurfaces spacelike? $w(\sigma) > 0$

BLACK HOLE + MATTER HALO

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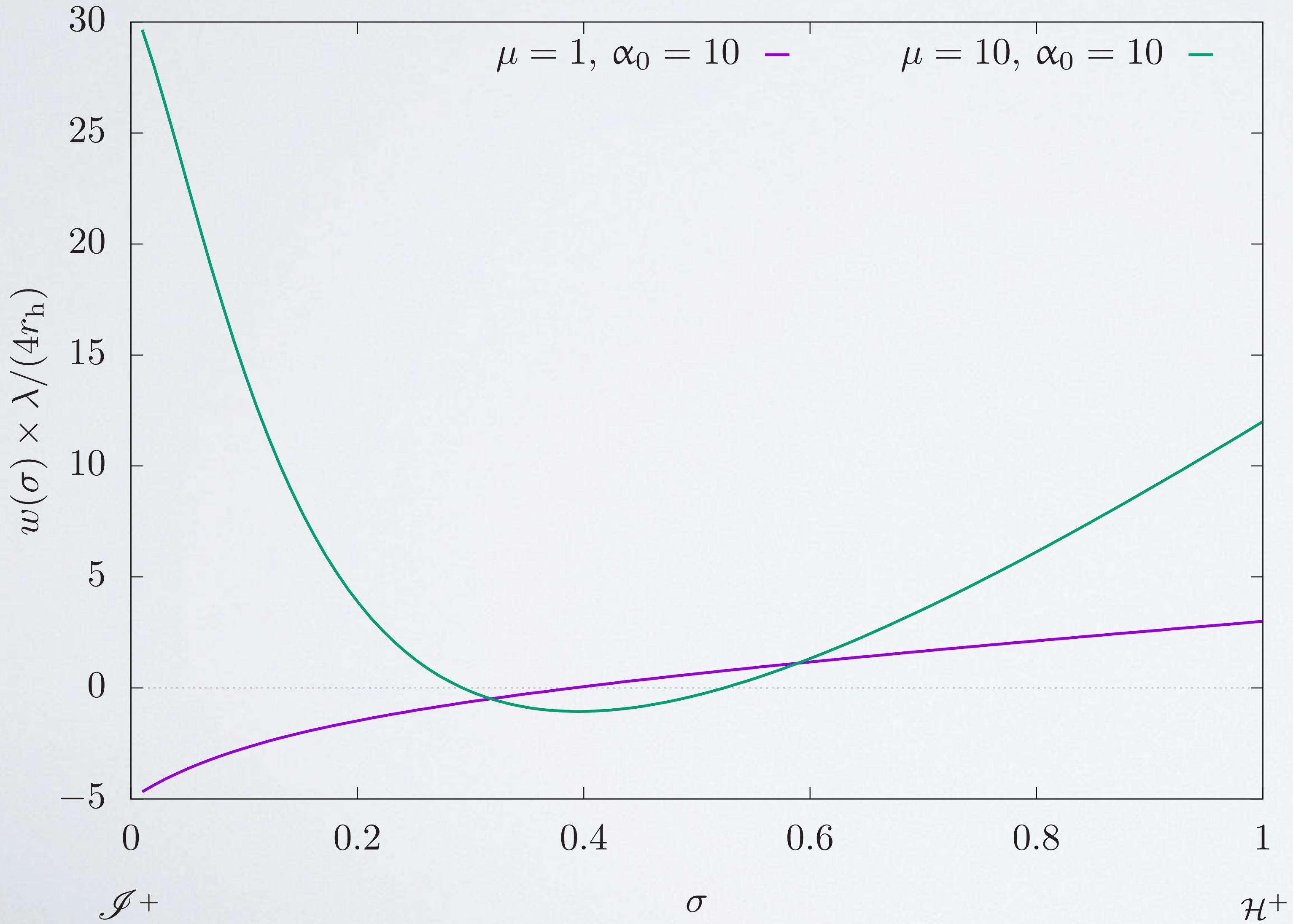

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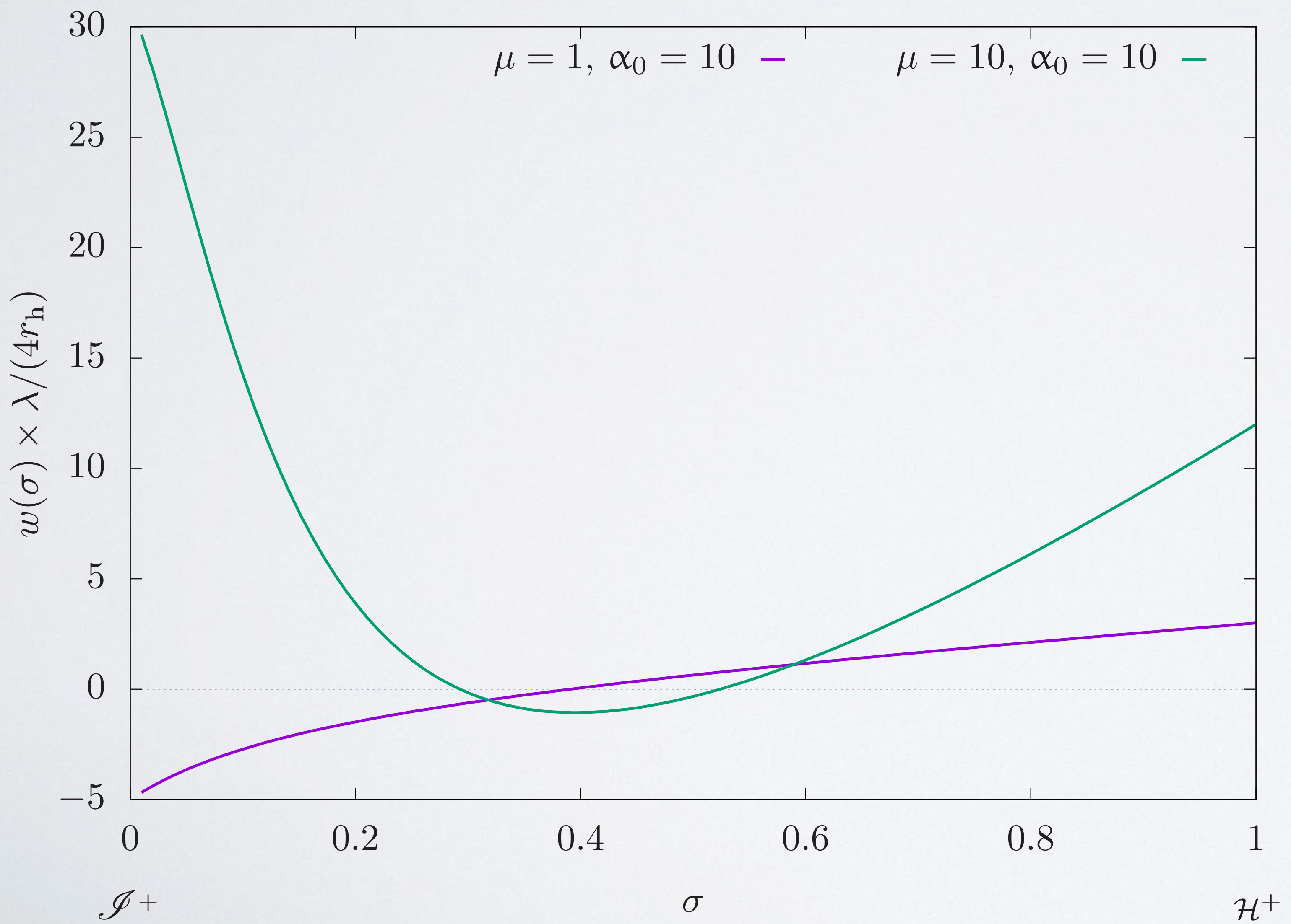

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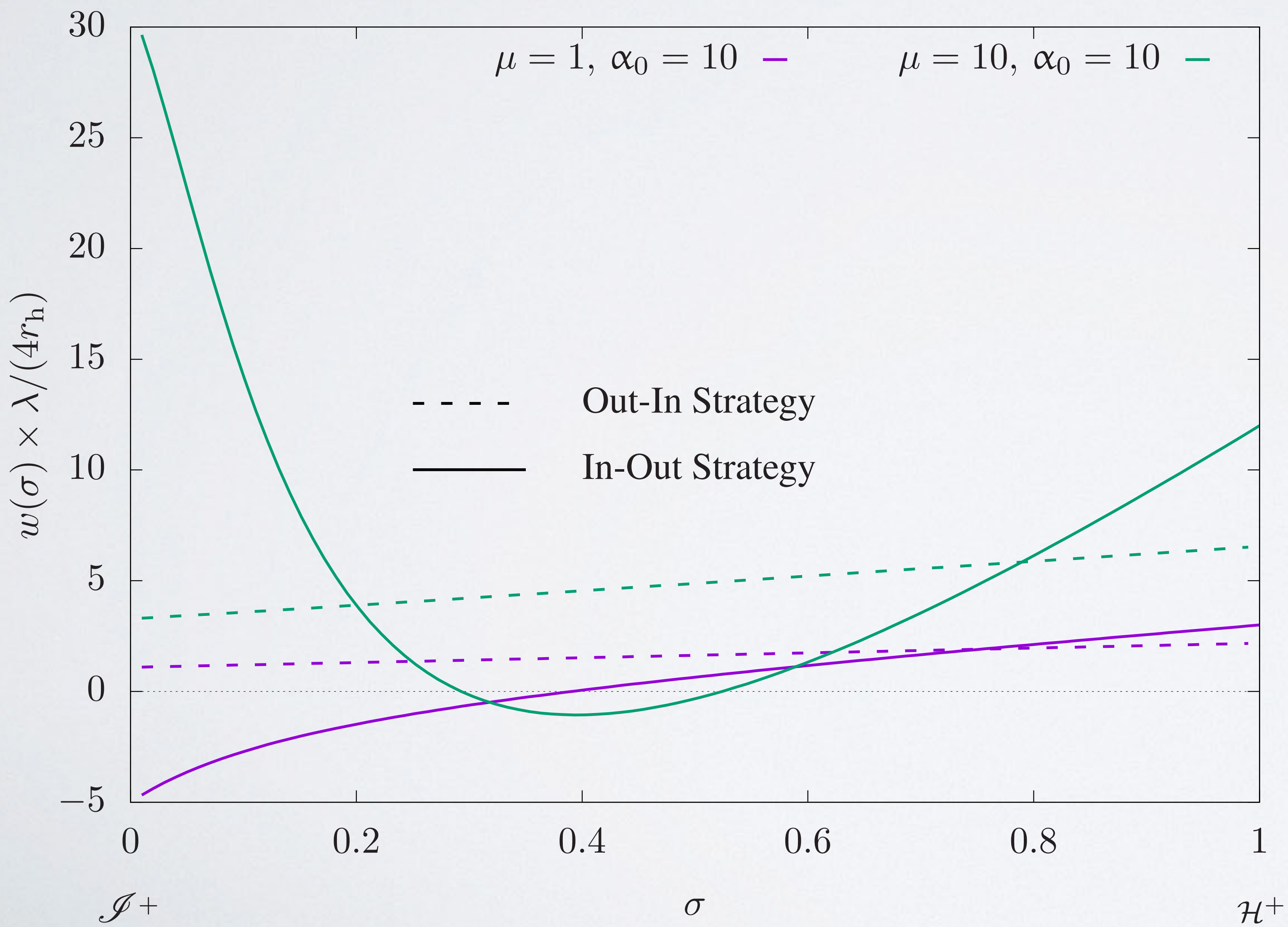

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(ArXiv 2307.15735)

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- Minimal gauge conceptual construction is based on coordinate change: there is no covariant classification to define it.

PROJECTS

Perturbation theory and Gravitational Wave astronomy

Second order: Self-force and ring-down
(e.g. regularity of source terms)

Environmental effects: coupling between waves with different characteristic speeds (regularity in time and frequency domain)

$$-\psi_{,tt} + \psi_{,r_* r_*} - V_\psi(r)\psi + C_\psi(r)\rho = 0$$

$$-v^{-2}\rho_{,tt} + \rho_{,r_* r_*} - V_\rho(r)\rho + C_\rho(r)\phi = 0$$

- Black-hole pertubation theory in the (regular) conformal set up: Derive Regge-Wheller-Zerrili and/or Bardeen-Press-Teukolsky equations *directly* from the conformal equations?

(So far, using coordinate trasformation, conformal re-scalings…)

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