Time across null horizons

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Nuggets of wisdom

- Horizon-penetrating time is hyperboloidal.
- Hyperboloidal coordinates are as natural in Lorentzian manifolds as spherical coordinates are in Riemannian.

Overview

Hyperboloids

Pseudo-spheres in Minkowski space; hyperbolic geometry

Horizons

Event horizon in Schwarzschild; cosmological horizon in de Sitter

Null infinity

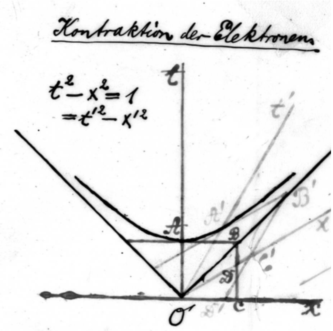
Spacelike regularization at infinity

Unification of space and time

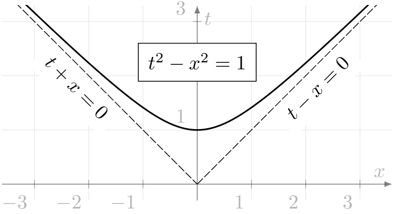
Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve independence.

Hermann Minkowski (1908)

The spacetime hyperbola

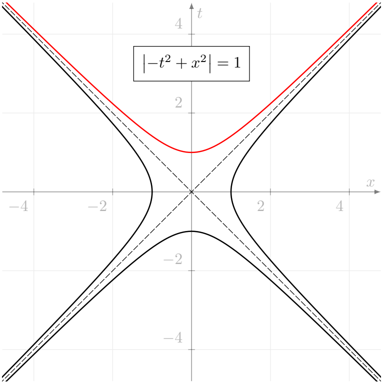


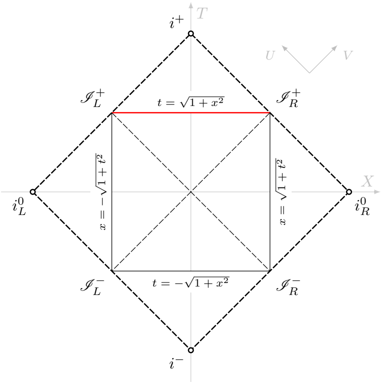
Unit spacetime hyperbola



Set of points at unit proper time from the origin.

Pseudo-circle





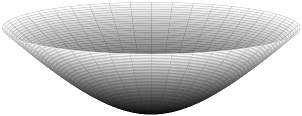
Hyperboloid model of hyperbolic geometry

Embed the spacetime hyperboloid, $t^2 - r^2 = 1$, in Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2,$$

by setting $t = \cosh\chi$ and $r = \sinh\chi$,

$$ds^2_{\mathbb{H}} = d\chi^2 + \sinh^2\chi d\theta^2.$$



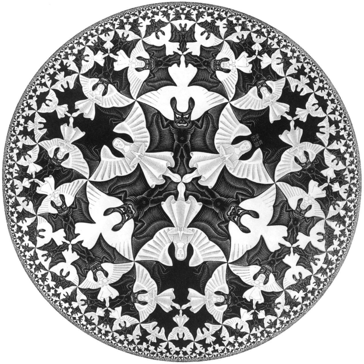
Hyperbolic tiling as a precursor to Penrose diagrams

Conformal disk model

$$t = \sqrt{1 + r^2}, \qquad r = \frac{2\rho}{1 - \rho^2}.$$

Escher's woodcuts (1959)

$$ds^2_{\mathbb{H}} = \frac{4}{(1 - \rho^2)^2} \left(d\rho^2 + \rho^2 d\theta^2 \right).$$



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Schwarzschild and Droste (1915–1917)

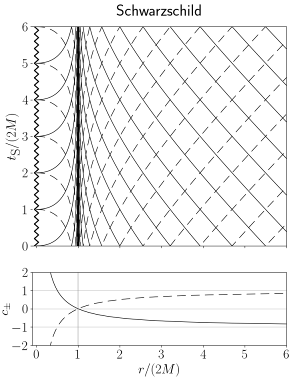
The Schwarzschild-Droste metric

$$ds^2 = -f\,dt^2 + \frac{1}{f}\,dr^2 + r^2 d\sigma^2,$$

with

$$f = 1 - \frac{2M}{r},$$

is singular at $r = 2M$.



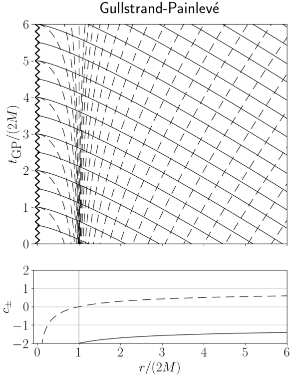
Gullstrand and Painlevé (1921–1922)

The Gullstrand-Painlevé metric

$$ds^2_{\mathrm{GP}} = -f dt^2_{\mathrm{GP}} + 2\sqrt{\frac{2M}{r}} dt_{\mathrm{GP}} dr + dr^2 + r^2 d\sigma^2.$$

is spatially flat with

$$t_{\mathrm{GP}} = t + \int \frac{\sqrt{1-f}}{f} dr.$$



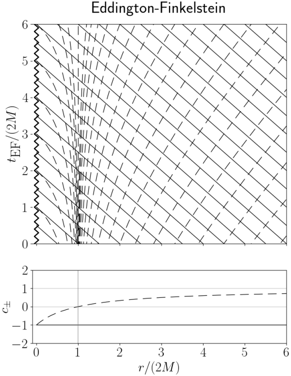
Eddington and Finkelstein (1924–1958)

The Eddington-Finkelstein metric

$$ds^2_{\mathrm{EF}} = -f dt^2_{\mathrm{EF}} + \frac{4M}{r} dt_{\mathrm{EF}} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\sigma^2,$$

is characteristic-preserving with

$$t_{\mathrm{EF}} + r = t + r_*, \quad r_* = \int \frac{dr}{f}.$$

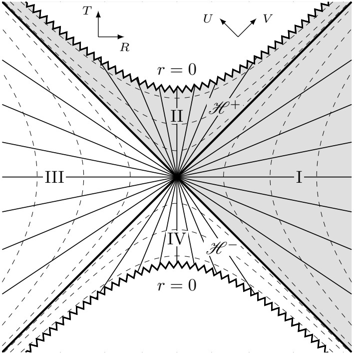


Kruskal and Szekeres (1960)

Regularize the "Schwarzschild singularity" using hyperboloidal coordinates,

$$T = e^{\frac{r}{4M}} \sqrt{\frac{r}{2M} - 1}\ \cosh \frac{t}{4M},$$

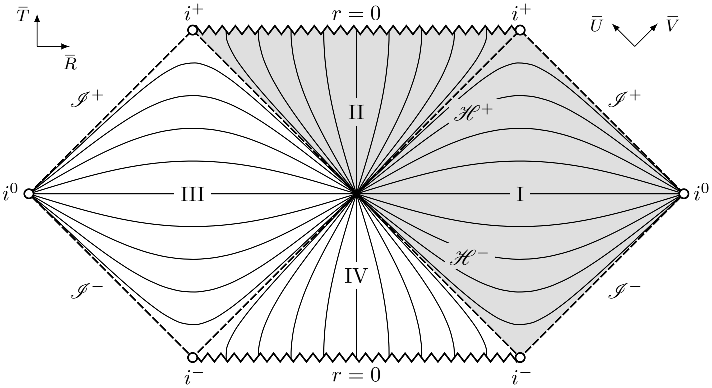
$$R = e^{\frac{r}{4M}} \sqrt{\frac{r}{2M} - 1}\ \sinh \frac{t}{4M}.$$



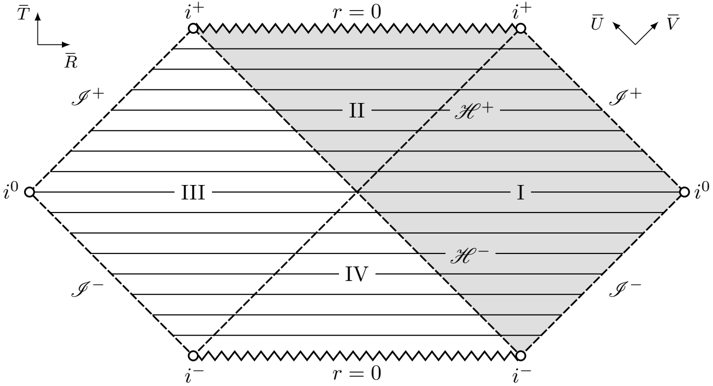
Level sets of r are hyperboloids.

$$T^2 - R^2 = e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1 \right).$$

Penrose (1963–1965)

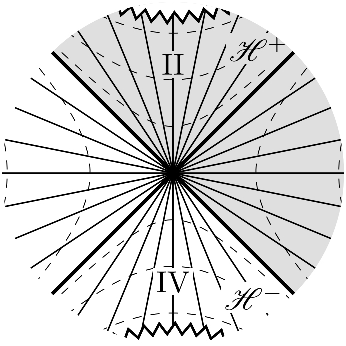


Penrose (1963–1965)

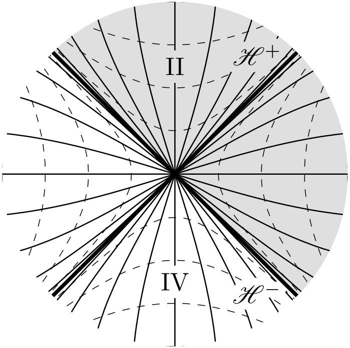


The bifurcation sphere is a coordinate singularity

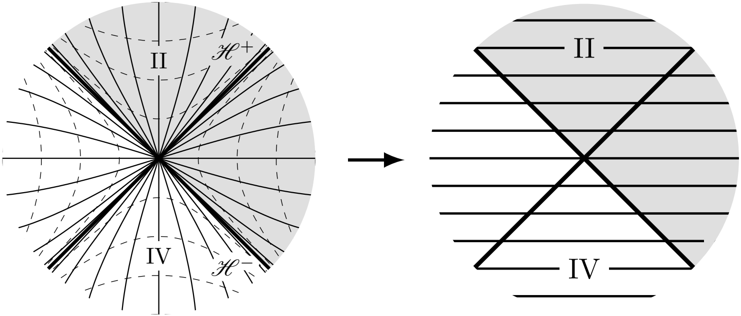
Kruskal



Penrose

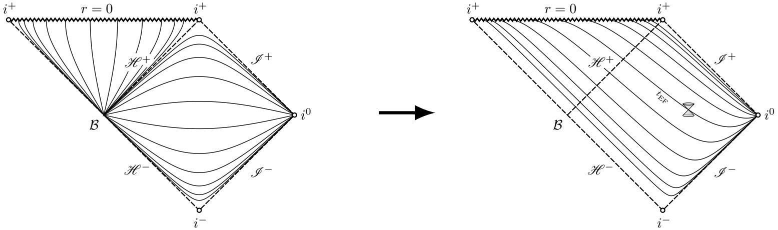


Resolved by horizon-penetrating coordinates



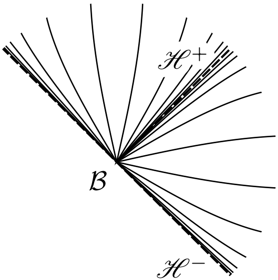
Penrose time is horizon-penetrating.

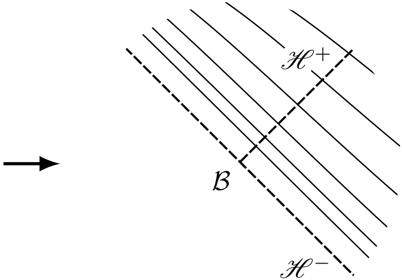
The transformation to regular coordinates is singular



$$t_{\rm EF} = t + 2M \ln\left(\frac{r}{2M} - 1\right).$$

The transformation to regular coordinates is singular





$$t_{\mathrm{EF}} = t + 2M \ln\left(\frac{r}{2M} - 1\right).$$

de Sitter spacetime (1917)

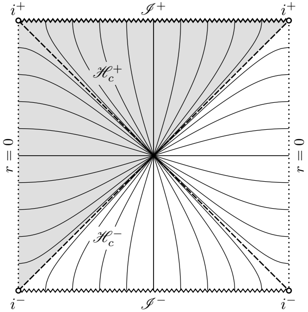
The de Sitter metric

$$ds^2_{\mathrm{dS}} = -f\,dt^2 + \frac{1}{f}\,dr^2 + r^2 d\sigma^2,$$

with

$$f = 1 - \frac{r^2}{L^2},$$

is singular at $r = L$.



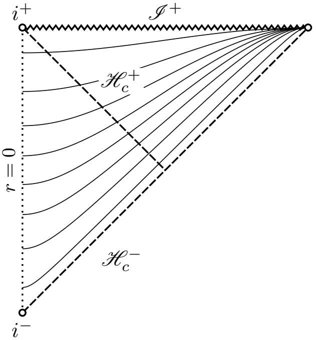
Spatially flat (Parikh, 2002)

The de Sitter metric,

$$ds_{\mathrm{dS}}^2 = -f\, dt_{\mathrm{P}}^2 - \frac{2r}{L} dt_{\mathrm{P}} dr + dr^2 + r^2 d\sigma^2.$$

is spatially flat with

$$t_{\mathrm{P}} = t + \int \frac{\sqrt{1-f}}{f} dr.$$



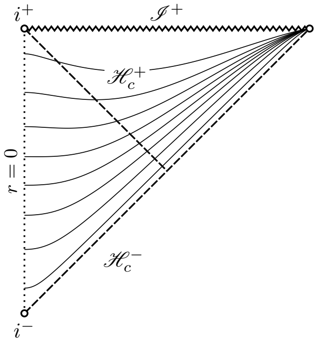
Characteristic-preserving (Misner, 2004)

The de Sitter metric,

$$ds^2_{\mathrm{dS}} = -f\, dt^2_{\mathrm{M}} - \frac{2r^2}{L^2} dt_{\mathrm{M}} dr + \left(1 + \frac{r^2}{L^2}\right) dr^2 + r^2 d\sigma^2.$$

is characteristic-preserving with

$$t_{\mathrm{M}} - r = t - r_*, \quad r_* = \int \frac{dr}{f}.$$



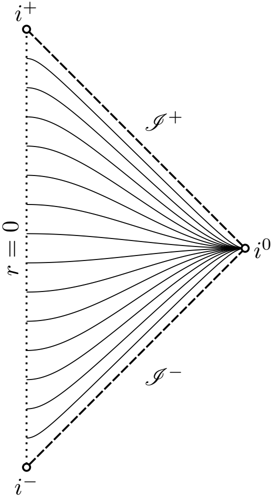
Overview

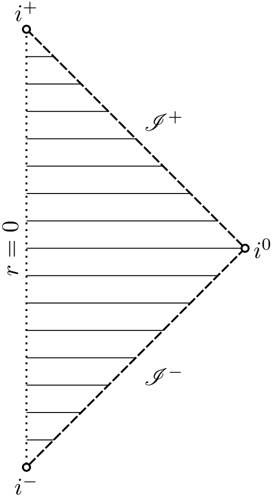
Hyperboloids
- Pseudo-spheres in Minkowski space; hyperbolic geometry

Horizons
- Event horizon in Schwarzschild; cosmological horizon in de Sitter

Null infinity
- Spacelike regularization at infinity

Spatial infinity is a coordinate singularity





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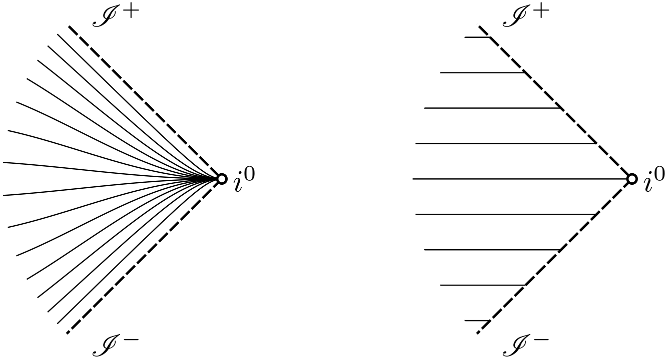


Milne Model (1933)

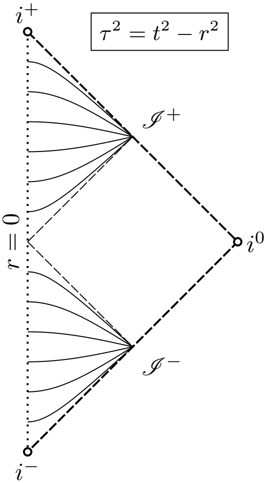
Milne coordinates satisfy

$$t^2 - r^2 = \tau^2.$$

Defining $r = \sinh\chi$, we get

$$ds^2 = -d\tau^2 + \tau^2 \left(d\chi^2 + \sinh^2\chi \, d\sigma^2\right).$$

Slices are hyperbolic spaces (Escher). The metric is time-dependent.



Hyperboloidal foliation (Moncrief, 2000)

Shift the unit hyperboloid along time

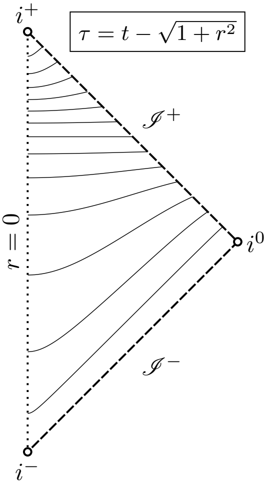
$$(t - \tau)^2 - r^2 = 1.$$

Again, with $r = \sinh\chi$, we get

$$ds^2 = -d\tau^2 - 2\sinh\chi\, d\tau\, d\chi + d\chi^2 + \sinh^2\chi\, d\sigma^2.$$

The metric is time-independent.

$$\partial_\tau = \partial_t.$$



Penrose coordinates

Penrose coordinates compactify null coordinates,

$$u = t - r = \tan U,$$

$$v = t + r = \tan V,$$

followed by decomposition into time and space

$$T = \frac{1}{2}(V - U), \qquad R = \frac{1}{2}(V + U).$$

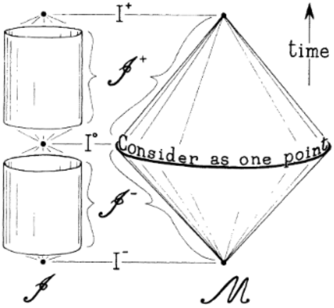


FIG. 1. Conformal structure of infinity.

Penrose time is hyperboloidal

Decompactified Penrose time,

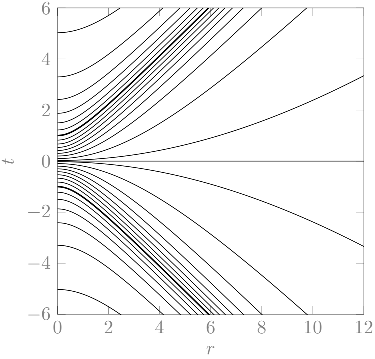
$$\tan(2T) = \frac{2t}{1 - t^2 + x^2}.$$

Defining $\tau \equiv -1/\tan(2T)$

$$(t - \tau)^2 - x^2 = 1 + \tau^2.$$

A combination of Milne time and hyperboloidal foliation.

The metric is time-dependent.



Penrose time is hyperboloidal

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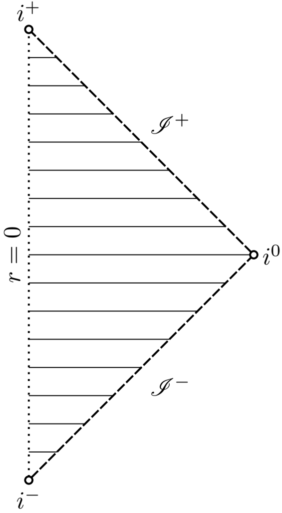


Scri-fixing with the timelike Killing field (AZ, 2008)

It is desirable to preserve the timelike Killing field, $\partial_\tau = \partial_t$ with,

$$\tau = t + h(r), \quad H(r) := \frac{dh}{dr}.$$

The metric is time-independent.

$$ds^2 = -f d\tau^2 + 2fH d\tau dr + \frac{1 - f^2 H^2}{f} dr^2 + r^2 d\Omega^2.$$

Compactify if needed,

$$\rho = g(r), \qquad G(\rho) := \frac{dg}{dr}.$$

The metric becomes

$$ds^2 = \frac{1}{G}\left(-fG d\tau^2 + 2fH d\tau d\rho + \frac{1-(fH)^2}{fG} d\rho^2 + G r^2 d\sigma^2\right).$$

Spatially flat or characteristic-preserving

One can show that the spatially flat gauge with

$$\frac{1 - f^2 H^2}{fG} = 1 \quad \& \quad Gr^2 = \rho^2,$$

and the characteristic-preserving gauge with

$$\tau \pm \rho = t \pm r_*,$$

are hyperboloidal.

Regular coordinates across null horizons are all hyperboloidal!

Minimal gauge (Ansorg and Macedo, 2013)

Future, spacelike regularity at the horizon

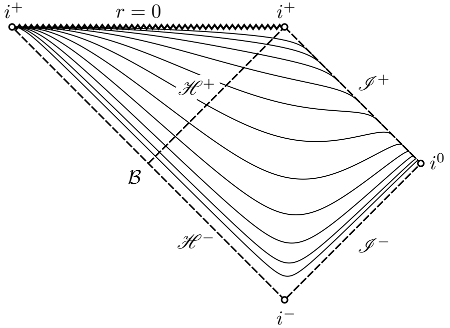
$$fH|_{r=2M} = 1,$$

and at infinity

$$fH|_{r\to\infty} \sim -1 + \frac{C}{r^2},$$

gives the minimal gauge,

$$fH = -1 + \frac{8M^2}{r^2}.$$



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